Online Supplemental Appendix to
"Threshold Regression with Nonparametric Sample Splitting"

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In Section S.1, this appendix contains the omitted proofs of technical lemmas used in the proofs of the main theorems. In Section S.2, it presents additional simulation results.

S.1 Omitted Proofs of Lemmas

Proof of Lemma A.1 For expositional simplicity, we present the case of scalar x_i. Let $\overline{g}$ be an integer satisfying $n^{(2+\varphi)/(2+2\varphi)}b_n\varpi/2 \leq \overline{g} \leq n^{(2+\varphi)/(2+2\varphi)}b_n\varpi$. Such a choice always exists since we assume $n^{(2+\varphi)/(2+2\varphi)}b_n\varpi \geq 1$. Consider a fine enough grid over $[\gamma_1, \gamma_1+\varpi]$ such that $\gamma_g = \gamma_1+(g-1)\varpi/\overline{g}$ for $g = 1, \ldots, \overline{g}+1$, where $\max_{1\leq g\leq \overline{g}}\left(\gamma_g - \gamma_{g-1}\right) \leq \varpi/\overline{g}$. We define $H_{ng}(s) = (nb_n)^{-1}\sum_{i\in\Lambda_n}|x_iu_iK_i(s)\mathbf{1}\left[\gamma_g < q_i \leq \gamma_{g+1}\right]|$ for $1 \leq g \leq \overline{g}$. Then for any $\gamma \in \left[\gamma_g, \gamma_{g+1}\right]$,

$$\left|J_n(\gamma; s) - J_n\left(\gamma_g; s\right)\right| \leq \sqrt{nb_n}\left|H_{ng}(s) - \mathbb{E}\left[H_{ng}(s)\right]\right| + \sqrt{nb_n}\mathbb{E}\left[H_{ng}(s)\right]$$

and hence

$$\begin{aligned}
& \sup_{\gamma\in[\gamma_1,\gamma_1+\varpi]}\left|J_n(\gamma; s) - J_n(\gamma_1; s)\right| \\
\leq\ & \max_{2\leq g\leq \overline{g}+1}\left|J_n\left(\gamma_g; s\right) - J_n(\gamma_1; s)\right| + \max_{1\leq g\leq \overline{g}}\sqrt{nb_n}\left|H_{ng}(s) - \mathbb{E}\left[H_{ng}(s)\right]\right| + \max_{1\leq g\leq \overline{g}}\sqrt{nb_n}\mathbb{E}\left[H_{ng}(s)\right] \\
\equiv\ & \Psi_1(s) + \Psi_2(s) + \Psi_3(s).
\end{aligned}$$

We let $h_i(s) = x_iu_iK_i(s)\mathbf{1}\left[\gamma_{g_1} < q_i \leq \gamma_{g_2}\right]$ for any given $1 \leq g_1 < g_2 \leq \overline{g}$ and for fixed $s \in \mathcal{S}_0$.

First, for $\Psi_1(s)$, we study

$$\begin{aligned}
& \mathbb{E}\left[\left|J_n\left(\gamma_{g_2}; s\right) - J_n\left(\gamma_{g_1}; s\right)\right|^4\right] \\
=\ & \frac{1}{n^2b_n^2}\sum_{i\in\Lambda_n}\mathbb{E}\left[h_i^4(s)\right] + \frac{1}{n^2b_n^2}\sum_{\substack{i,j\in\Lambda_n\\ i\neq j}}\mathbb{E}\left[h_i^2(s)h_j^2(s)\right] + \frac{1}{n^2b_n^2}\sum_{\substack{i,j\in\Lambda_n\\ i\neq j}}\mathbb{E}\left[h_i^3(s)h_j(s)\right] \\
& + \frac{1}{n^2b_n^2}\sum_{\substack{i,j,k,l\in\Lambda_n\\ i\neq j\neq k\neq l}}\mathbb{E}\left[h_i(s)h_j(s)h_k(s)h_l(s)\right] + \frac{1}{n^2b_n^2}\sum_{\substack{i,j,k\in\Lambda_n\\ i\neq j\neq k}}\mathbb{E}\left[h_i^2(s)h_j(s)h_k(s)\right] \\
\equiv\ & \Psi_{11}(s) + \Psi_{12}(s) + \Psi_{13}(s) + \Psi_{14}(s) + \Psi_{15}(s),
\end{aligned}$$

where each term's bound is obtained as follows.

For $\Psi_{11}(s)$*:* From Lemma B.1 below, we have $b_n^{-1}\mathbb{E}\left[h_i^4(s)\right] \leq C|\gamma_{g_2} - \gamma_{g_1}|$ for $C < \infty$. Therefore,

$$\Psi_{11}(s) \leq \frac{C}{nb_n}\left|\gamma_{g_2} - \gamma_{g_1}\right| = \frac{C}{nb_n\left|\gamma_{g_2} - \gamma_{g_1}\right|}\left(\gamma_{g_2} - \gamma_{g_1}\right)^2 \leq C_1\left(\gamma_{g_2} - \gamma_{g_1}\right)^2$$

for some $C_1 < \infty$ with sufficiently large n, where the last inequality is because

$$|\gamma_{g_2} - \gamma_{g_1}| \leq |g_2 - g_1| (\varpi/\overline{g}) = O\left((n^{(2+\varphi)/(2+2\varphi)} b_n)^{-1}\right) \tag{B.1}$$

by construction and hence $nb_n|\gamma_{g_2} - \gamma_{g_1}| = O(n^{\varphi/(2+2\varphi)})$ grows with n.

For $\Psi_{12}(s)$: Note that

$$\begin{aligned}
&\frac{1}{n^2 b_n^2} \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \left| Cov\left[h_i^2(s), h_j^2(s)\right]\right| \qquad \text{(B.2)} \\
\leq_{[1]}\ & \frac{C_2}{n^2 b_n^2} \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \alpha_{1,1}(\lambda(i,j))^{\varphi/(2+\varphi)} \mathbb{E}\left[\left|h_i^2(s)\right|^{2+\varphi}\right]^{2/(2+\varphi)} \\
\leq_{[2]}\ & \frac{C_2}{n^2 b_n^2} \mathbb{E}\left[\left|h_i^2(s)\right|^{2+\varphi}\right]^{2/(2+\varphi)} \sum_{i \in \Lambda_n} \sum_{m=1}^{n-1} \sum_{\substack{j \in \Lambda_n \\ \lambda(i,j) \in [m, m+1)}} \alpha_{1,1}(m)^{\varphi/(2+\varphi)} \\
\leq_{[3]}\ & \frac{C_2}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \left(\frac{1}{b_n} \mathbb{E}\left[\left|h_i^2(s)\right|^{2+\varphi}\right]\right)^{2/(2+\varphi)} \sum_{i \in \Lambda_n} \sum_{m=1}^{\infty} m \alpha_{1,1}(m)^{\varphi/(2+\varphi)} \\
\leq_{[4]}\ & \frac{C_2'}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \left(C\left|\gamma_{g_2} - \gamma_{g_1}\right|\right)^{2/(2+\varphi)} n \sum_{m=1}^{\infty} m \exp\left(-m\varphi/(2+\varphi)\right) \\
\leq_{[5]}\ & \frac{C_2''}{n b_n^{(2+2\varphi)/(2+\varphi)}} \left|\gamma_{g_2} - \gamma_{g_1}\right|^{-(2+2\varphi)/(2+\varphi)} \left(\gamma_{g_2} - \gamma_{g_1}\right)^2 \\
\leq_{[6]}\ & C_2''' \left(\gamma_{g_2} - \gamma_{g_1}\right)^2
\end{aligned}$$

for some $C_2, C_2', C_2'', C_2''' < \infty$, where ineq.[1] is by the covariance inequality (A.1) with $p_x = 2+\varphi$, $q_x = 2+\varphi$, $r_x = (2+\varphi)/\varphi$, and $k_x = l_x = 1$; ineq.[2] follows by dividing the observations according to $\lambda(i,j)$; ineq.[3] follows from that $\alpha_{1,1}(\lambda(i,j)) \leq \alpha_{1,1}(m)$ for $\lambda(i,j) \in [m, m+1)$ and $|\{j \in \Lambda_n : \lambda(i,j) \in [m, m+1)\}| = O(m)$ for any given $i \in \Lambda_n$ as in Lemma A.1.(iii) of Jenish and Prucha (2009); ineq.[4] is by Lemma B.1; ineq.[5] follows from $\sum_{m=1}^{\infty} m \exp\left(-m\varphi/(2+\varphi)\right) < \infty$ for $\varphi > 0$; and ineq.[6] follows from that $|g_2 - g_1|^{2/(2+\varphi)} \leq (g_2 - g_1)^2$ and $nb_n^{(2+2\varphi)/(2+\varphi)}|\gamma_{g_2} - \gamma_{g_1}|^{(2+2\varphi)/(2+\varphi)} = O(1)$ from (B.1). Furthermore, by Lemma B.1, we have $b_n^{-1}\mathbb{E}\left[h_i^2(s)\right] \leq C|\gamma_{g_2} - \gamma_{g_1}|$ and thus Assumptions A-(iii), (v), and (x) yield that

$$\begin{aligned}
\Psi_{12}(s) &\leq \frac{1}{n^2 b_n^2} \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \left(\mathbb{E}\left[h_i^2(s)\right] \mathbb{E}\left[h_j^2(s)\right] + \left|Cov\left[h_i^2(s), h_j^2(s)\right]\right|\right) \qquad \text{(B.3)} \\
&\leq \left(\frac{1}{b_n} \mathbb{E}\left[h_i^2(s)\right]\right)^2 + \frac{1}{n^2 b_n^2} \sum_{\substack{i,j \in \Lambda_n \\ i \neq j}} \left|Cov\left[h_i^2(s), h_j^2(s)\right]\right| \\
&\leq \left(C^2 + C_2'''\right) \left(\gamma_{g_2} - \gamma_{g_1}\right)^2
\end{aligned}$$

from (B.2).

For $\Psi_{13}(s)$: Since $\mathbb{E}\left[h_i(s)\right] = 0$, using the same argument as (B.2) and (B.3), and the inequality (A.1) with $p_x = 2(2+\varphi)/3$, $q_x = 2(2+\varphi)$, $r_x = (2+\varphi)/\varphi$, and $k_x = l_x = 1$, we can also show that

$$\begin{aligned}
\Psi_{13}(s) &\leq \frac{1}{n^2 b_n^2} \sum_{\substack{i,j\in\Lambda_n \\ i\neq j}} \left| Cov\left[h_i^3(s), h_j(s)\right]\right| \\
&\leq \frac{C_3}{n^2 b_n^2} \sum_{\substack{i,j\in\Lambda_n \\ i\neq j}} \alpha_{1,1}(\lambda(i,j))^{\varphi/(2+\varphi)} \mathbb{E}\left[\left|h_i^3(s)\right|^{2(2+\varphi)/3}\right]^{3/(4+2\varphi)} \mathbb{E}\left[\left|h_j(s)\right|^{2(2+\varphi)}\right]^{1/(4+2\varphi)} \\
&\leq \frac{C_3'}{n b_n^{(2+2\varphi)/(2+\varphi)}} \left(\frac{1}{b_n}\mathbb{E}\left[\left|h_i^2(s)\right|^{(2+\varphi)}\right]\right)^{2/(2+\varphi)} \sum_{m=1}^{\infty} m \exp\left(-m\varphi/(2+\varphi)\right) \\
&\leq C_3''(n b_n^{(2+2\varphi)/(2+\varphi)})^{-1} \left|\gamma_{g_2} - \gamma_{g_1}\right|^{2/(2+\varphi)} \\
&\leq C_3''' \left(\gamma_{g_2} - \gamma_{g_1}\right)^2
\end{aligned}$$

for $C_3, C_3', C_3'', C_3''' < \infty$.

For $\Psi_{14}(s)$*:* Let $\mathcal{E} = \{(i,j,k,l) : i \neq j \neq k \neq l,\ 1 < \lambda(i,j) \leq \lambda(i,k) \leq \lambda(i,l)$, and $\lambda(j,k) \leq \lambda(j,l)\}$.[11] Then by stationarity,

$$\begin{aligned}
\Psi_{14}(s) &\leq \frac{4!}{n^2 b_n^2} \sum_{i,j,k,l\in\Lambda_n\cap\mathcal{E}} \left|\mathbb{E}\left[h_i(s)h_j(s)h_k(s)h_l(s)\right]\right| \\
&= \frac{2\cdot 4!}{n^2 b_n^2} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(i,j)\geq\max\{\lambda(j,k),\lambda(k,l)\}}} \left|Cov\left[h_i(s), \{h_j(s)h_k(s)h_l(s)\}\right]\right| \\
&\quad + \frac{4!}{n^2 b_n^2} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\}}} \left|Cov\left[\{h_i(s)h_j(s)\}, \{h_k(s)h_l(s)\}\right]\right| \\
&\quad + \frac{4!}{n^2 b_n^2} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\}}} \left|\mathbb{E}\left[h_i(s)h_j(s)\right]\mathbb{E}\left[h_k(s)h_l(s)\right]\right| \\
&\equiv \Psi_{14,1}(s) + \Psi_{14,2}(s) + \Psi_{14,3}(s).
\end{aligned}$$

In $\Psi_{14,1}(s)$, note that the largest distance among all the pairs is $\lambda(i,j)$. Then, similarly, by the covariance inequality (A.1) with $p_x = 2(2+\varphi)/3$, $q_x = 2(2+\varphi)$, $r_x = (2+\varphi)/\varphi$, $k_x = 1$, and $l_x = 3$,

$$\begin{aligned}
&\Psi_{14,1}(s) \\
&\leq \frac{C_4}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(i,j)\geq\max\{\lambda(j,k),\lambda(k,l)\}}} \alpha_{1,3}(\lambda(i,j))^{\varphi/(2+\varphi)} \left(\frac{1}{b_n}\mathbb{E}\left[\left|h_i(s)\right|^{4+2\varphi}\right]\right)^{1/(4+2\varphi)} \\
&\qquad \times \left(\frac{1}{b_n}\mathbb{E}\left[\left|h_j(s)h_k(s)h_l(s)\right|^{2(2+\varphi)/3}\right]\right)^{3/(4+2\varphi)} \\
&\leq \frac{C_4'}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(i,j)\geq\max\{\lambda(j,k),\lambda(k,l)\}}} \alpha_{1,3}(\lambda(i,j))^{\varphi/(2+\varphi)} \left|\gamma_{g_2} - \gamma_{g_1}\right|^{2/(2+\varphi)}
\end{aligned}$$

[11] In the (one-dimensional) time series case, this set of indices reduces to $\{(i,j,k,l) : 1 \leq i < j < k < l \leq n\}$.

$$\begin{aligned}
&\leq \frac{C_4'}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{i\in\Lambda_n} \sum_{m=1}^{n-1} \sum_{\substack{j\in\Lambda_n \\ \lambda(i,j)\in[m,m+1)}} \sum_{\substack{k\in\Lambda_n \\ \lambda(j,k)\leq m}} \sum_{\substack{l\in\Lambda_n \\ \lambda(k,l)\leq m}} \alpha_{1,3}(m)^{\varphi/(2+\varphi)} \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \\
&\leq \frac{C_4''}{n b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{m=1}^{\infty} m^5 \exp\left(-m\varphi/(2+\varphi)\right) \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \\
&\leq C_4''' \left(\gamma_{g_2}-\gamma_{g_1}\right)^2
\end{aligned}$$

as in (B.2) for $C_4, C_4', C_4'', C_4''' < \infty$ since $|\{k \in \Lambda_n : \lambda(j,k) \leq m\}| = O(m^2)$ for any given $j \in \Lambda_n$. Note that the second inequality above is from

$$\begin{aligned}
\mathbb{E}[|h_j(s)h_k(s)h_l(s)|^{2(2+\varphi)/3}] &\leq \left(\mathbb{E}[|h_j(s)h_k(s)|^{2(2+\varphi)/2}]\right)^{2/3} \left(\mathbb{E}[|h_l(s)|^{2(2+\varphi)}]\right)^{1/3} \\
&\leq \left(\mathbb{E}[|h_j(s)|^{2(2+\varphi)}]\mathbb{E}[|h_k(s)|^{2(2+\varphi)}]\right)^{1/3} \left(\mathbb{E}[|h_l(s)|^{2(2+\varphi)}]\right)^{1/3} \\
&= \mathbb{E}[|h_j(s)|^{2(2+\varphi)}]
\end{aligned}$$

by the Hölder's inequality and stationarity. In $\Psi_{14,2}(s)$, the largest distance among all the pairs is $\lambda(j,k)$. Similarly as above, therefore,

$$\begin{aligned}
&\Psi_{14,2}(s) \\
&\leq \frac{C_5}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\}}} \alpha_{2,2}(\lambda(j,k))^{\varphi/(2+\varphi)} \left(\frac{1}{b_n}\mathbb{E}\left[|h_i(s)h_j(s)|^{2+\varphi}\right]\right)^{1/(2+\varphi)} \\
&\qquad \times \left(\frac{1}{b_n}\mathbb{E}\left[|h_k(s)h_l(s)|^{2+\varphi}\right]\right)^{1/(2+\varphi)} \\
&\leq \frac{C_5'}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\}}} \alpha_{2,2}(\lambda(j,k))^{\varphi/(2+\varphi)} \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \\
&\leq \frac{C_5''}{n b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{m=1}^{\infty} m^5 \exp\left(-m\varphi/(2+\varphi)\right) \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \\
&\leq C_5''' \left(\gamma_{g_2}-\gamma_{g_1}\right)^2
\end{aligned}$$

for $C_5, C_5', C_5'', C_5''' < \infty$. In $\Psi_{14,3}(s)$, the largest distance among all the pairs is still $\lambda(j,k)$. We define an increasing sequence of integers κ_n such that $\kappa_n^2 = O(n^{(2+\varphi)/(2+2\varphi)})$. We decompose $\Psi_{14,3}(s)$ into

$$\begin{aligned}
\Psi_{14,3}(s) &= \frac{4!}{n^2 b_n^2} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\} \\ \lambda(i,j)\leq\kappa_n,\lambda(k,l)\leq\kappa_n}} \left|\mathbb{E}\left[h_i(s)h_j(s)\right]\mathbb{E}\left[h_k(s)h_l(s)\right]\right| \\
&\quad + \frac{4!}{n^2 b_n^2} \sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E} \\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\} \\ \lambda(i,j)>\kappa_n,\lambda(k,l)>\kappa_n}} \left|\mathbb{E}\left[h_i(s)h_j(s)\right]\mathbb{E}\left[h_k(s)h_l(s)\right]\right|
\end{aligned}$$

$$+\frac{2\cdot 4!}{n^2 b_n^2}\sum_{\substack{i,j,k,l\in\Lambda_n\cap\mathcal{E}\\ \lambda(j,k)\geq\max\{\lambda(i,j),\lambda(k,l)\}\\ \lambda(i,j)\leq\kappa_n,\lambda(k,l)>\kappa_n}}\left|\mathbb{E}\left[h_i(s)h_j(s)\right]\mathbb{E}\left[h_k(s)h_l(s)\right]\right|$$

$$\equiv\quad \Psi_{14,3}^{[1]}(s)+\Psi_{14,3}^{[2]}(s)+\Psi_{14,3}^{[3]}(s).$$

For $\Psi_{14,3}^{[1]}(s)$, since $\mathbb{E}\left[|x_i u_i x_j u_j||q_i,q_j,s_i,s_j\right]<C_6<\infty$ from Assumption A-(vii), we can show that

$$\begin{aligned}\frac{1}{b_n^2}\mathbb{E}\left[h_i(s)h_j(s)\right] &\leq C_6\iint\int_{\gamma_g}^{\gamma_k}\int_{\gamma_g}^{\gamma_k}K(t)\,K(t')\,f\left(q,q',s+tb_n,s+t'b_n\right)dqdq'dtdt' \qquad \text{(B.4)}\\ &\leq C_6'\left(\gamma_{g_2}-\gamma_{g_1}\right)^2\end{aligned}$$

for some constant $C_6'<\infty$ when n is sufficiently large, using a similar argument in the proof of Lemma B.1. Hence, from the fact that $|\{j\in\Lambda_n:\lambda(i,j)\leq\kappa_n\}|=O(\kappa_n^2)$ for any fixed $i\in\Lambda_n$, we obtain

$$\begin{aligned}\Psi_{14,3}^{[1]}(s) &\leq \frac{C_6''}{n^2b_n^2}\sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)\leq\kappa_n}}\left|\mathbb{E}\left[h_i(s)h_j(s)\right]\right|\sum_{\substack{k,l\in\Lambda_n\\ \lambda(k,l)\leq\kappa_n}}\left|\mathbb{E}\left[h_k(s)h_l(s)\right]\right|\\ &\leq C_6'''\kappa_n^4 b_n^2\left(\gamma_{g_2}-\gamma_{g_1}\right)^4\\ &\leq C_6^*\left(\gamma_{g_2}-\gamma_{g_1}\right)^2\end{aligned}$$

for some $C_6'',C_6''',C_6^*<\infty$, because $\kappa_n^4 b_n^2(\gamma_{g_2}-\gamma_{g_1})^2=O((\kappa_n^2 n^{-(2+\varphi)/(2+2\varphi)})^2)=O(1)$ from the construction of κ_n. For $\Psi_{14,3}^{[2]}(s)$, since $\mathbb{E}\left[h_i(s)h_j(s)\right]=Cov\left[h_i(s),h_j(s)\right]$, the covariance inequality (A.1) and Lemma B.1 yield that

$$\begin{aligned}\Psi_{14,3}^{[2]}(s) &\leq \frac{C_7}{n^2b_n^2}\sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)>\kappa_n}}\left|\mathbb{E}\left[h_i(s)h_j(s)\right]\right|\sum_{\substack{k,l\in\Lambda_n\\ \lambda(k,l)>\kappa_n}}\left|\mathbb{E}\left[h_k(s)h_l(s)\right]\right|\\ &\leq \frac{C_7'}{n^2b_n^2}\left\{\sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)>\kappa_n}}\alpha_{1,1}\left(\lambda\left(i,j\right)\right)^{\varphi/(2+\varphi)}\mathbb{E}\left[\left|h_i(s)\right|^{2+\varphi}\right]^{1/(2+\varphi)}\mathbb{E}\left[\left|h_j(s)\right|^{2+\varphi}\right]^{1/(2+\varphi)}\right\}^2\\ &= \frac{C_7'}{b_n^{2\varphi/(2+\varphi)}}\left\{\left(\frac{1}{b_n}\mathbb{E}\left[\left|h_i(s)\right|^{2+\varphi}\right]\right)^{2/(2+\varphi)}\frac{1}{n}\sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)>\kappa_n}}\alpha_{1,1}\left(\lambda\left(i,j\right)\right)^{\varphi/(2+\varphi)}\right\}^2\\ &\leq \frac{C_7''}{b_n^{2\varphi/(2+\varphi)}}\left|\gamma_{g_2}-\gamma_{g_1}\right|^{4/(2+\varphi)}\left\{\frac{1}{n}\sum_{i\in\Lambda_n}\sum_{m=\kappa_n+1}^{n-1}\sum_{\substack{j\in\Lambda_n\\ \lambda(i,j)\in[m,m+1)}}\alpha_{1,1}\left(m\right)^{\varphi/(2+\varphi)}\right\}^2\\ &\leq \frac{C_7'''}{b_n^{2\varphi/(2+\varphi)}}\left|\gamma_{g_2}-\gamma_{g_1}\right|^{4/(2+\varphi)}\left\{\sum_{m=\kappa_n+1}^{\infty}m\exp\left(-m\varphi/(2+\varphi)\right)\right\}^2\end{aligned}$$

for $C_7, C_7', C_7'', C_7''' < \infty$. Note that for any $a > 0$, we have

$$\sum_{m=\kappa_n+1}^{\infty} m \exp(-am) \leq \int_{\kappa_n}^{\infty} t \exp(-at)\, dt = \frac{1}{a}\left(\kappa_n + \frac{1}{a}\right) \exp(-a\kappa_n).$$

It follows that

$$\begin{aligned}
\Psi_{14,3}^{[2]}(s) &\leq C_7^* \left(\gamma_{g_2} - \gamma_{g_1}\right)^2 \left(\frac{\kappa_n^2 \exp\left(-2\kappa_n \varphi/(2+\varphi)\right)}{b_n^{2\varphi/(2+\varphi)} \left|\gamma_k - \gamma_g\right|^{2\varphi/(2+\varphi)}}\right) \\
&\leq C_7^{**} \left(\gamma_{g_2} - \gamma_{g_1}\right)^2 \exp\left(-\frac{2\varphi}{2+\varphi}\kappa_n\right) \left(\frac{\kappa_n^2}{n^{(2+\varphi)/(2+2\varphi)}}\right) n^{(2-\varphi)/(2+2\varphi)} \\
&\leq C_7^{***} \left(\gamma_{g_2} - \gamma_{g_1}\right)^2
\end{aligned}$$

for $C_7^*, C_7^{**}, C_7^{***} < \infty$, because $b_n^{2\varphi/(2+\varphi)} |\gamma_{g_2} - \gamma_{g_1}|^{2\varphi/(2+\varphi)} = O(n^{2\varphi/(2+2\varphi)})$ and the exponential term decays faster than the (potentially) growing polynomial term. For $\Psi_{14,3}^{[3]}(s)$, by combining the arguments for bounding $\Psi_{14,3}^{[1]}(s)$ and $\Psi_{14,3}^{[2]}(s)$, we also obtain that

$$\begin{aligned}
\Psi_{14,3}^{[3]}(s) &\leq C_8 \left(\kappa_n^2 b_n \left(\gamma_{g_2} - \gamma_{g_1}\right)^2\right) \left(\left|\gamma_{g_2} - \gamma_{g_1}\right|^{2/(2+\varphi)} \frac{\kappa_n \exp(-\kappa_n \varphi/(2+\varphi))}{b_n^{\varphi/(2+\varphi)}}\right) \\
&\leq C_8' \left(\gamma_{g_2} - \gamma_{g_1}\right)^2
\end{aligned}$$

for some $C_8, C_8' < \infty$.

For $\Psi_{15}(s)$: Let $\mathcal{E}' = \{(i,j,k) : i \neq j \neq k \text{ and } 1 < \lambda(i,j) \leq \lambda(i,k)\}$ and decompose it into

$$\begin{aligned}
\Psi_{15}(s) &= \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k \in \Lambda_n \cap \mathcal{E}' \\ \lambda(i,j) < \lambda(j,k)}} \mathbb{E}\left[h_i^2(s) h_j(s) h_k(s)\right] + \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k \in \Lambda_n \cap \mathcal{E}' \\ \lambda(i,j) \geq \lambda(j,k)}} \mathbb{E}\left[h_i^2(s) h_j(s) h_k(s)\right] \\
&\leq \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k \in \Lambda_n \cap \mathcal{E}' \\ \lambda(i,j) < \lambda(j,k)}} \left|Cov\left[\{h_i^2(s) h_j(s)\}, h_k(s)\right]\right| \\
&\quad + \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k \in \Lambda_n \cap \mathcal{E}' \\ \lambda(i,j) \geq \lambda(j,k)}} \left|Cov\left[h_i^2(s), \{h_j(s) h_k(s)\}\right]\right| \\
&\quad + \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k \in \Lambda_n \cap \mathcal{E}' \\ \lambda(i,j) \geq \lambda(j,k)}} \left|\mathbb{E}\left[h_i^2(s)\right] \mathbb{E}\left[h_j(s) h_k(s)\right]\right| \\
&\equiv \Psi_{15,1}(s) + \Psi_{15,2}(s) + \Psi_{15,3}(s).
\end{aligned}$$

Similarly as $\Psi_{14,1}(s)$,

$$\begin{aligned}
&\Psi_{15,1}(s) \\
\leq\; & \frac{C_9}{n^2 b_n^2} \sum_{\substack{i,j,k \in \Lambda_n \cap \mathcal{E}' \\ \lambda(i,j) < \lambda(j,k)}} \alpha_{2,1}\left(\lambda(j,k)\}\right)^{\varphi/(2+\varphi)} \mathbb{E}\left[\left|h_i^2(s) h_j(s)\right|^{2(2+\varphi)/3}\right]^{3/(4+2\varphi)} \mathbb{E}\left[\left|h_k(s)\right|^{2(2+\varphi)}\right]^{1/(4+2\varphi)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C_9'}{n^2 b_n^{(2+2\varphi)/(2+\varphi)}} \sum_{j\in\Lambda_n} \sum_{m=1}^{n-1} \sum_{\substack{k\in\Lambda_n \\ \lambda(j,k)\in[m,m+1)}} \sum_{\substack{i\in\Lambda_n \\ \lambda(i,j)\leq m}} \alpha_{2,1}(m)^{\varphi/(2+\varphi)} \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \\
&\leq \frac{C_9''}{n b_n^{(2+2\varphi)/(2+\varphi)}} \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \sum_{m=1}^{\infty} m^3 \exp\left(-m\varphi/(2+\varphi)\right) \\
&\leq C_9''' \left(\gamma_{g_2}-\gamma_{g_1}\right)^2,
\end{aligned}
$$

for $C_9, C_9', C_9'', C_9''' < \infty$ and the same argument implies that $\Psi_{15,2}(s) = O((\gamma_{g_2}-\gamma_{g_1})^2)$ as well. For $\Psi_{15,3}(s)$, similarly as $\Psi_{14,3}(s)$, we have

$$
\begin{aligned}
\Psi_{15,3}(s) &= \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k\in\Lambda_n\cap\mathcal{E}' \\ \lambda(i,j)\geq\lambda(j,k),\lambda(j,k)\leq\kappa_n}} \left|\mathbb{E}\left[h_i^2(s)\right]\mathbb{E}\left[h_j(s)h_k(s)\right]\right| \\
&\quad + \frac{2}{n^2 b_n^2} \sum_{\substack{i,j,k\in\Lambda_n\cap\mathcal{E}' \\ \lambda(i,j)\geq\lambda(j,k),\lambda(j,k)>\kappa_n}} \left|\mathbb{E}\left[h_i^2(s)\right]\mathbb{E}\left[h_j(s)h_k(s)\right]\right| \\
&\leq \frac{C_{10}}{n b_n} \sum_{i\in\Lambda_n} \left|\mathbb{E}\left[h_i^2(s)\right]\right| \left\{ \frac{1}{n b_n} \sum_{\substack{j,k\in\Lambda_n \\ \lambda(j,k)\leq\kappa_n}} \left|\mathbb{E}\left[h_j(s)h_k(s)\right]\right| + \frac{1}{n b_n} \sum_{\substack{j,k\in\Lambda_n \\ \lambda(j,k)>\kappa_n}} \left|\mathbb{E}\left[h_j(s)h_k(s)\right]\right| \right\} \\
&\leq C_{10}' \left|\gamma_{g_2}-\gamma_{g_1}\right| \left\{ \kappa_n^2 b_n \left(\gamma_{g_2}-\gamma_{g_1}\right)^2 + \left|\gamma_{g_2}-\gamma_{g_1}\right|^{2/(2+\varphi)} \frac{\kappa_n \exp\left(-\kappa_n(\varphi/(2+\varphi))\right)}{b_n^{\varphi/(2+\varphi)}} \right\} \\
&\leq C_{10}'' \left(\gamma_{g_2}-\gamma_{g_1}\right)^2 \left\{ \kappa_n^2 b_n \left|\gamma_{g_2}-\gamma_{g_1}\right| + \frac{\kappa_n \exp\left(-\kappa_n(\varphi/(2+\varphi))\right)}{b_n^{\varphi/(2+\varphi)} \left|\gamma_{g_2}-\gamma_{g_1}\right|^{\varphi/(2+\varphi)}} \right\} \\
&\leq C_{10}''' \left(\gamma_{g_2}-\gamma_{g_1}\right)^2
\end{aligned}
$$

for $C_{10}, C_{10}', C_{10}'', C_{10}''' < \infty$.

By combining all the results from $\Psi_{11}(s)$ to $\Psi_{15}(s)$, we thus have

$$
\mathbb{E}\left[\left|J_n\left(\gamma_{g_2};s\right) - J_n\left(\gamma_{g_1};s\right)\right|^4\right] \leq \overline{C}\left(\gamma_{g_2}-\gamma_{g_1}\right)^2 \leq \overline{C}\left(\left|g_2-g_1\right|(\varpi/\overline{g})\right)^2
$$

for $\overline{C} < \infty$, and hence Theorem 12.2 of Billingsley (1968) yields that there exists $\overline{C}' < \infty$ such that

$$
\mathbb{P}\left(\Psi_1(s) > \eta\right) = \mathbb{P}\left(\max_{2\leq g\leq\overline{g}+1} \left|J_n\left(\gamma_g;s\right) - J_n\left(\gamma_1;s\right)\right| > \eta\right) \leq \frac{\overline{C}'\varpi^2}{\eta^4} \tag{B.5}
$$

for $\eta > 0$ as in the proof of Lemma A.3 in Hansen (2000).

Next, for $\Psi_2(s)$, for all $g = 1, \ldots, \bar{g}$ and $\overline{C}'' < \infty$, Lemma B.2 below shows that

$$
\mathbb{E}\left[\left(\sqrt{n b_n}\left|H_{ng}(s) - \mathbb{E}\left[H_{ng}(s)\right]\right|\right)^4\right] \leq \overline{C}''(\varpi/\overline{g})^2.
$$

Hence, by Markov inequality,

$$
\mathbb{P}\left(\max_{1\leq g\leq\overline{g}} \sqrt{n b_n}\left|H_{ng}(s) - \mathbb{E}\left[H_{ng}(s)\right]\right| > \eta\right) \leq \overline{g}\frac{\overline{C}''\varpi^2/\overline{g}^2}{\eta^4} \leq \frac{\overline{C}''\varpi^2}{\eta^4}, \tag{B.6}
$$

where the last inequality uses $(\varpi/\bar{g}) \leq \varpi$.

Finally, for $\Psi_3(s)$, Lemma B.1 gives

$$
\begin{aligned}
\sqrt{nb_n}\mathbb{E}\left[H_{ng}(s)\right] &= \sqrt{nb_n} \times b_n^{-1}\mathbb{E}\left[\left|x_i u_i \mathbf{1}\left[\gamma_g < q_i \leq \gamma_{g+1}\right] K_i\left(s\right)\right|\right] \\
&\leq \sqrt{nb_n} C \left|\gamma_{g+1} - \gamma_g\right| \leq \frac{C^{**}\sqrt{nb_n}}{n^{(2+\varphi)/(2+2\varphi)}b_n} = \frac{C^{**}}{\sqrt{n^{1/(1+\varphi)}b_n}} \leq \eta
\end{aligned}
\tag{B.7}
$$

if there exists a constant C^{**} such that $\eta \geq C^{**}(n^{1/(1+\varphi)}b_n)^{-1/2}$. So the proof is complete by combining (B.5), (B.6), and (B.7), where $C^* = \overline{C}' + \overline{C}''$. ■

The following two lemmas are used in proving Lemma A.1 above.

Lemma B.1 $b_n^{-1}\mathbb{E}\left[|h_i(s)|^\ell\right] \leq C\left|\gamma_{g_2} - \gamma_{g_1}\right|$ *for* $\ell \leq 2(2+\varphi)$ *and* $C < \infty$.

Proof of Lemma B.1 We have $\mathbb{E}\left[|x_i u_i|^\ell | q_i, s_i\right] < C_1 < \infty$ from Assumption A-(v). Hence, by Assumptions A-(vii) and (x), Taylor expansion yields

$$
\begin{aligned}
\frac{1}{b_n}\mathbb{E}\left[|h_i(s)|^\ell\right] &= \frac{1}{b_n}\iint \mathbb{E}\left[|x_i u_i|^\ell | q, v\right] \mathbf{1}\left[\gamma_{g_1} < q \leq \gamma_{g_2}\right] K^\ell\left(\frac{v-s}{b_n}\right) f\left(q,v\right) dqdv \\
&\leq C_1 \iint \mathbf{1}\left[\gamma_{g_1} < q \leq \gamma_{g_2}\right] K^\ell\left(t\right) f\left(q, s + b_n t\right) dqdt \\
&= C_1 \int K^\ell\left(t\right) \int \mathbf{1}\left[\gamma_{g_1} < q \leq \gamma_{g_2}\right] \left\{f\left(q,s\right) + O(b_n t + b_n^2 t^2)\right\} dqdt \\
&\leq C_1' \left|\gamma_{g_2} - \gamma_{g_1}\right|
\end{aligned}
$$

for some constants $C_1, C_1' < \infty$ when n is sufficiently large, where we apply the change of variables $t = (v-s)/b_n$. Note that $\int K^\ell\left(t\right) dt < \infty$ and $\int \mathbf{1}\left[\gamma_{g_1} < q \leq \gamma_{g_2}\right] f\left(q,s\right) dq = f_s(s)\mathbb{P}(\gamma_{g_1} < q_i \leq \gamma_{g_2} | s_i = s) = O(|\gamma_{g_2} - \gamma_{g_1}|)$ by the mean-value theorem, where $f_s\left(s\right) < \infty$ is the marginal density of s_i and $|\gamma_{g_2} - \gamma_{g_1}| \leq |g_2 - g_1|\varpi/\bar{g} = O(n^{(2+\varphi)/(2+2\varphi)}b_n)^{-1} = o(1)$ because $n^{(2+\varphi)/(2+2\varphi)}b_n > n^{1/(1+\varphi)}b_n \to \infty$ as $n \to \infty$ from Assumption A-(ix). ■

Lemma B.2 $\mathbb{E}[(\sqrt{nb_n}\,|H_{ng}(s) - \mathbb{E}[H_{ng}(s)]|)^4] \leq C(\varpi/\bar{g})^2$ *for all* $g = 1, \ldots, \bar{g}$ *and* $C < \infty$.

Proof of Lemma B.2 Recall $H_{ng}(s) = (nb_n)^{-1}\sum_{i\in\Lambda_n}|h_{ig}(s)|$, where $h_{ig}(s) = x_i u_i K_i\left(s\right)\mathbf{1}[\gamma_g < q_i \leq \gamma_{g+1}]$. We decompose

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sqrt{nb_n}\left|H_{ng}(s) - \mathbb{E}\left[H_{ng}(s)\right]\right|\right)^4\right] \qquad \text{(B.8)} \\
=\ & \frac{1}{n^2 b_n^2}\mathbb{E}\left[\left(\sum_{i\in\Lambda_n}\left(\left|h_{ig}\left(s\right)\right| - \mathbb{E}\left[\left|h_{ig}(s)\right|\right]\right)\right)^4\right] \\
=\ & \frac{1}{n^2 b_n^2}\sum_{i\in\Lambda_n}\mathbb{E}\left[\eta_i(s)^4\right] + \frac{1}{n^2 b_n^2}\sum_{\substack{i,j\in\Lambda_n \\ i\neq j}}\mathbb{E}\left[\eta_i^2(s)\eta_j^2\left(s\right)\right] + \frac{1}{n^2 b_n^2}\sum_{\substack{i,j\in\Lambda_n \\ i\neq j}}\mathbb{E}\left[\eta_i^3\left(s\right)\eta_j\left(s\right)\right] \\
&+ \frac{1}{n^2 b_n^2}\sum_{\substack{i,j,k,l\in\Lambda_n \\ i\neq j\neq k\neq l}}\mathbb{E}\left[\eta_i(s)\eta_j\left(s\right)\eta_k\left(s\right)\eta_l\left(s\right)\right] + \frac{1}{n^2 b_n^2}\sum_{\substack{i,j,k\in\Lambda_n \\ i\neq j\neq k}}\mathbb{E}\left[\eta_i^2\left(s\right)\eta_j\left(s\right)\eta_k\left(s\right)\right],
\end{aligned}
$$

where we define $\eta_i(s) = |h_{ig}(s)| - \mathbb{E}[|h_{ig}(s)|]$. However, $\mathbb{E}[\eta_i(s)] = 0$ by construction, $\mathbb{E}[|\eta_i(s)|^r] \leq \mathbb{E}[|h_{ig}(s)|^r]$ for any $r \geq 1$, and $\mathbb{E}[|\eta_i(s)\eta_j(s)|] \leq \mathbb{E}[|h_{ig}(s) h_{jg}(s)|]$. It follows that, using the same arguments respectively in Lemma B.1, (B.2), and (B.4), we obtain

$$\begin{aligned} \frac{1}{b_n}\mathbb{E}\left[|\eta_i(s)|^\ell\right] &\leq \frac{1}{b_n}\mathbb{E}\left[|h_{ig}(s)|^\ell\right] \leq C_1 |\gamma_{g+1} - \gamma_g|, \\ \frac{1}{n^2 b_n^2} \sum_{\substack{i,j \in \Lambda_n \\ i, \neq j}} \left|Cov\left[\eta_i^2(s), \eta_j^2(s)\right]\right| &\leq C_2 \left(\gamma_{g+1} - \gamma_g\right)^2 \\ \frac{1}{b_n^2}\mathbb{E}\left[|\eta_i(s)\eta_j(s)|\right] &\leq \frac{1}{b_n^2}\mathbb{E}\left[|h_{ig}(s) h_{jg}(s)|\right] \leq C_3 \left(\gamma_{g+1} - \gamma_g\right)^2 \end{aligned}$$

for sufficiently large n and for some $C_1, C_2, C_3 < \infty$. Therefore, we can find the bounds of the five terms in (B.8) as we obtain for $\Psi_{1k}(s)$ in the proof of Lemma A.1 by replacing (g_1, g_2) by $(g, g+1)$ for $k = 1, \ldots, 5$, from the facts that $|\gamma_{g+1} - \gamma_g| = \varpi/\bar{g}$. Note that a similar moment inequality for spatial α-mixing processes can be also found in Gao, Lu, and Tjøstheim (2008). ■

Proof of Lemma A.2 For a fixed γ, the Theorem of Bolthausen (1982) implies that $J_n(\gamma; s) \to_d J(\gamma; s)$ as $n \to \infty$ under Assumption A-(iii). Because γ is in the indicator function, such pointwise convergence in γ can be generalized into any finite collection of γ to yield the finite dimensional convergence in distribution. Then the weak convergence follows from Lemma A.1 above and Theorem 15.5 of Billingsley (1968). ■

Proof of Lemma A.3 We prove the convergence of $M_n(\gamma; s)$. For $J_n(\gamma; s)$, since $\mathbb{E}[u_i x_i | q_i, s_i] = 0$, the proof is identical as $M_n(\gamma; s)$ and hence omitted. For expositional simplicity, we present the case of scalar x_i.

By stationarity, Assumptions A-(vii), (x), and Taylor expansion, we have

$$\begin{aligned} \mathbb{E}[M_n(\gamma; s)] &= \frac{1}{b_n} \iint \mathbb{E}[x_i^2 | q, v] \mathbf{1}[q \leq \gamma] K\left(\frac{v - s}{b_n}\right) f(q, v)\, dq dv \qquad \text{(B.9)} \\ &= \iint D(q, s + b_n t) \mathbf{1}[q \leq \gamma] K(t) f(q, s + b_n t)\, dq dt \\ &= M(\gamma; s) + b_n^2 \int \widetilde{M}(q; s) \mathbf{1}[q \leq \gamma] dq \int t^2 K(t)\, dt, \end{aligned}$$

where $\widetilde{M}(q; s) = \dot{D}(q, s)\dot{f}(q, s) + (\ddot{D}(q, s) + \ddot{f}(q, s))/2$. We let $\dot{D}$ and $\dot{f}$ denote the partial derivatives, and $\ddot{D}$ and $\ddot{f}$ denote the second-order partial derivatives with respect to s. Since $\sup_{s \in \mathcal{S}_0} ||\widetilde{M}(q; s)|| < \infty$ for any q from Assumption A-(vii), and $K(\cdot)$ is a second-order kernel, we have

$$\sup_{(\gamma, s) \in \Gamma \times \mathcal{S}_0} \|\mathbb{E}[M_n(\gamma; s)] - M(\gamma; s)\| = O_p\left(b_n^2\right) = o_p(1). \qquad \text{(B.10)}$$

Next, we let $\tau_n = (n \log n)^{1/(4+2\varphi)}$ and $\varphi > 0$ be given in Assumption A-(v). By Markov's and Hölder's inequalities, Assumption A-(v) gives $\mathbb{P}\left(x_n^2 > \tau_n\right) \leq C\tau_n^{-(4+2\varphi)} \mathbb{E}[|x_n^2|^{2(2+\varphi)}] \leq C' (n \log n)^{-1}$ for some $C, C' < \infty$. Thus

$$\sum_{n \in \mathbb{Z}^2} \mathbb{P}\left(x_n^2 > \tau_n\right) \leq C' \sum_{n \in \mathbb{Z}^2} (n \log n)^{-1} < \infty,$$

which yields that $x_n^2 \leq \tau_n$ almost surely for sufficiently large n by the Borel-Cantelli lemma. Since $\tau_n \to \infty$ as $n \to \infty$, we have $x_i^2 \leq \tau_n$ for any $i \in \Lambda_n$ and hence

$$\sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n(\gamma;s) - M_n^\tau(\gamma;s)\| = 0 \quad \text{and} \quad \sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|\mathbb{E}[M_n(\gamma;s)] - \mathbb{E}[M_n^\tau(\gamma;s)]\| = 0$$

almost surely for sufficiently large n, where

$$M_n^\tau(\gamma;s) = \frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2 \mathbf{1}_i(\gamma)K_i(s)\mathbf{1}\left\{x_i^2 \leq \tau_n\right\}. \tag{B.11}$$

It follows that

$$\begin{aligned}\sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n(\gamma;s) - \mathbb{E}[M_n(\gamma;s)]\| \;\leq\; & \sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n(\gamma;s) - M_n^\tau(\gamma;s)\| \\ & + \sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n^\tau(\gamma;s) - \mathbb{E}[M_n^\tau(\gamma;s)]\| \\ & + \sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|\mathbb{E}[M_n(\gamma;s)] - \mathbb{E}[M_n^\tau(\gamma;s)]\|\end{aligned} \tag{B.12}$$

and we establish $\sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n(\gamma;s) - \mathbb{E}[M_n(\gamma;s)]\| = o_p(1)$ if the second term in (B.12) is $o_p(1)$. Then we conclude $\sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n(\gamma;s) - M(\gamma;s)\| \to_p 0$ as desired by combining (B.10) and (B.12).

To this end, we let m_n be an integer such that $m_n = O(\tau_n(n/(b_n^3 \log n))^{1/2})$ and we cover the compact $\Gamma \times \mathcal{S}_0$ by m_n^2 squares centered at $\left(\gamma_{k_1}, s_{k_2}\right)$, defined as $\mathcal{I}_k = \{(\gamma', s') : |\gamma' - \gamma_{k_1}| \leq C/m_n$ and $|s' - s_{k_2}| \leq C/m_n\}$ for some $C < \infty$. Note that $\tau_n(n/(b_n^3 \log n))^{1/2} = \tau_n \left(n^{1-2\epsilon} b_n/\log n\right)^{1/2} \left(n^{2\epsilon}/b_n^4\right)^{1/2} \to \infty$ as $n \to \infty$ from Assumption A-(xi), hence $m_n \to \infty$. We then have

$$\begin{aligned}\sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0} \|M_n^\tau(\gamma;s) - \mathbb{E}[M_n^\tau(\gamma;s)]\| \;\leq\; & \max_{\substack{1\leq k_1\leq m_n \\ 1\leq k_2\leq m_n}} \sup_{(\gamma,s)\in\mathcal{I}_k} \|M_n^\tau(\gamma;s) - \mathbb{E}[M_n^\tau(\gamma;s)]\| \\ \leq\; & \max_{\substack{1\leq k_1\leq m_n \\ 1\leq k_2\leq m_n}} \sup_{(\gamma,s)\in\mathcal{I}_k} \left\|M_n^\tau(\gamma;s) - M_n^\tau(\gamma_{k_1};s_{k_2})\right\| \\ & + \max_{\substack{1\leq k_1\leq m_n \\ 1\leq k_2\leq m_n}} \sup_{(\gamma,s)\in\mathcal{I}_k} \left\|\mathbb{E}[M_n^\tau(\gamma;s)] - \mathbb{E}\left[M_n^\tau(\gamma_{k_1};s_{k_2})\right]\right\| \\ & + \max_{\substack{1\leq k_1\leq m_n \\ 1\leq k_2\leq m_n}} \left\|M_n^\tau(\gamma_{k_1};s_{k_2}) - \mathbb{E}\left[M_n^\tau(\gamma_{k_1};s_{k_2})\right]\right\| \\ \equiv\; & \Psi_{M1} + \Psi_{M2} + \Psi_{M3}.\end{aligned}$$

We first decompose $M_n^\tau(\gamma;s) - M_n^\tau(\gamma_{k_1};s_{k_2}) \leq M_{1n}^\tau(\gamma,\gamma_{k_1};s_{k_2}) + M_{2n}^\tau(\gamma;s,s_{k_2})$, where

$$\begin{aligned}M_{1n}^\tau(\gamma,\gamma_{k_1};s_{k_2}) &= \frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2 \left|\mathbf{1}[q_i \leq \gamma] - \mathbf{1}\left[q_i \leq \gamma_{k_1}\right]\right| K_i(s_{k_2})\mathbf{1}\left[x_i^2 \leq \tau_n\right], \\ M_{2n}^\tau(\gamma;s,s_{k_2}) &= \frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2 \mathbf{1}[q_i \leq \gamma] \left|K_i(s) - K_i(s_{k_2})\right| \mathbf{1}\left[x_i^2 \leq \tau_n\right].\end{aligned}$$

Since $K_i(\cdot)$ is bounded from Assumption A-(x) and we only consider $x_i^2 \leq \tau_n$, for any γ such that

$|\gamma - \gamma_{k_1}| \leq C/m_n$,

$$\begin{aligned}
\left\|M_{1n}^{\tau}(\gamma,\gamma_{k_1};s_{k_2})\right\| &\leq C_1\frac{\tau_n}{nb_n}\sum_{i\in\Lambda_n}\mathbf{1}\left[\min\{\gamma_{k_1},\gamma\} < q_i \leq \max\{\gamma_{k_1},\gamma\}\right] \qquad \text{(B.13)}\\
&\leq C_1\tau_n b_n^{-1}\mathbb{P}\left(\min\{\gamma_{k_1},\gamma\} < q_i \leq \max\{\gamma_{k_1},\gamma\}\right)(1+o_{a.s.}(1))\\
&\leq C_1'\tau_n b_n^{-1}m_n^{-1}(1+o_{a.s.}(1))\\
&= C_1''\left(\frac{b_n\log n}{n}\right)^{1/2}(1+o_{a.s.}(1))\\
&= O_{a.s.}\left(\left(\frac{\log n}{nb_n}\right)^{1/2}\right)
\end{aligned}$$

for some $C_1, C_1', C_1'' < \infty$, where the second equality is by the uniform almost sure law of large numbers for random fields (e.g., Jenish and Prucha (2009), Theorem 2). This bound holds uniformly in $(\gamma, s) \in \mathcal{I}_k$ and $k_1, k_2 \in \{1, \ldots, m_n\}$. Similarly, since $K(\cdot)$ is Lipschitz from Assumption A-(x),

$$\begin{aligned}
\|M_{2n}^{\tau}(\gamma;s,s_{k_2})\| &\leq \frac{\tau_n}{nb_n}\sum_{i\in\Lambda_n}|K_i(s)-K_i(s_{k_2})| \qquad \text{(B.14)}\\
&\leq C_2\frac{\tau_n}{b_n^2}|s-s_{k_2}| \leq \frac{C_2'\tau_n}{b_n^2 m_n} = O_{a.s.}\left(\left(\frac{\log n}{nb_n}\right)^{1/2}\right)
\end{aligned}$$

for some $C_2, C_2' < \infty$, uniformly in γ, s, k_1 and k_2. It follows that

$$\left\|M_n^{\tau}(\gamma;s) - M_n^{\tau}(\gamma_{k_1};s_{k_2})\right\| = O_{a.s.}((\log n/(nb_n))^{1/2})$$

uniformly in γ, s, k_1 and k_2, and hence we can readily verify that both Ψ_{M1} and Ψ_{M2} are $O_{a.s.}((\log n/(nb_n))^{1/2})$. For Ψ_{M3}, we follow the same argument for bounding the Q_{3n}^* term on pp.794-796 of Carbon, Francq, and Tran (2007). In particular, for any $k_1 \in \{1, \ldots, m_n\}$, $\max_{1\leq k_2\leq m_n} ||M_n^{\tau}(\gamma_{k_1};s_{k_2}) - \mathbb{E}\left[M_n^{\tau}(\gamma_{k_1};s_{k_2})\right]|| \leq C_3(\log n/(nb_n))^{1/2}$ a.s. for some $C_3 < \infty$. Note that γ_{k_1} shows up in the indicator function $\mathbf{1}\left[q_i \leq \gamma_{k_1}\right]$ only, which is uniformly bounded by 1. The bound is hence uniform over all $k_1 \in \{1, \ldots, m_n\}$ and $\Psi_{M3} = O_{a.s.}((\log n/(nb_n))^{1/2})$ as well. We have $\sup_{(\gamma,s)\in\Gamma\times\mathcal{S}_0}\|M_n^{\tau}(\gamma;s) - \mathbb{E}\left[M_n^{\tau}(\gamma;s)\right]\| = o_{a.s.}(1)$ by combining the bounds for Ψ_{M1}, Ψ_{M2}, and Ψ_{M3}. We thus complete the proof because $\log n/(nb_n) \to 0$ from Assumption A-(ix). ■

Proof of Lemma A.4 For expositional simplicity, we present the case of scalar x_i. Similarly as (B.9), we have

$$\begin{aligned}
&\mathbb{E}\left[\Delta M_n(s)\right] \qquad \text{(B.15)}\\
=& \iint \mathcal{D}(q,s+b_nt)\left\{\mathbf{1}\left[q<\gamma_0(s+b_nt)\right] - \mathbf{1}\left[q<\gamma_0(s)\right]\right\}K(t)dqdt\\
=& \int_{\mathcal{T}^+(s)}\int_{\gamma_0(s)}^{\gamma_0(s+b_nt)}\mathcal{D}(q,s+b_nt)K(t)dqdt + \int_{\mathcal{T}^-(s)}\int_{\gamma_0(s+b_nt)}^{\gamma_0(s)}\mathcal{D}(q,s+b_nt)K(t)dqdt\\
\equiv& \Psi_M^+(s) + \Psi_M^-(s),
\end{aligned}$$

where $\mathcal{D}(q, s+b_n t) = D(q, s+b_n t) f(q, s+b_n t)$ and we define $\mathcal{T}^+(s) = \{t : \gamma_0(s) \leq \gamma_0(s+b_n t)\}$ and $\mathcal{T}^-(s) = \{t : \gamma_0(s) > \gamma_0(s+b_n t)\}$. We consider three cases of $\dot{\gamma}_0(s) = \partial\gamma_0(s)/\partial s > 0$, $\dot{\gamma}_0(s) < 0$, and $\dot{\gamma}_0(s) = 0$ separately, which are well-defined from Assumption A-(vi).

First, we suppose $\dot{\gamma}_0(s) > 0$. We choose a positive sequence $t_n \to \infty$ such that $t_n b_n \to 0$ as $n \to \infty$. It follows that for any fixed $\varepsilon > 0$, $t_n b_n \leq \varepsilon$ if n is sufficiently large and hence $\mathcal{T}^+(s) \cap \{t : |t| \leq t_n\} = [0, t_n]$ since $\dot{\gamma}_0(\cdot)$ is continuous. Furthermore, the mean value theorem gives

$$\int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t) dq = b_n t \mathcal{D}(\gamma_0(s), s) \dot{\gamma}_0(s) + O\left(b_n^2\right) t^2, \tag{B.16}$$

where $|\mathcal{D}(\gamma_0(s), s)\dot{\gamma}_0(s)| < \infty$ from Assumptions A-(vi) and (vii). Therefore,

$$\begin{aligned} & \Psi_M^+(s) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{B.17}) \\ = \ & \int_0^{t_n} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t) K(t) dq dt + \int_{\mathcal{T}^+(s)\cap\{t:|t|>t_n\}} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t) K(t) dq dt \\ = \ & \left\{ b_n \mathcal{D}(\gamma_0(s), s) \dot{\gamma}_0(s) \int_0^{t_n} t K(t) dt + O(b_n^2) \right\} + O\left(b_n \int_{t_n}^{\infty} t K(t) dt \right) \\ = \ & b_n \mathcal{D}(\gamma_0(s), s) \dot{\gamma}_0(s) \int_0^{\infty} t K(t) dt + o(b_n), \end{aligned}$$

where the second equality is because $\mathcal{T}^+(s) \cap \{t : |t| > t_n\} \subset (t_n, \infty)$ and

$$\left| \int_{\mathcal{T}^+(s)\cap\{t:|t|>t_n\}} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t) K(t) dq dt \right| \leq b_n \left|\mathcal{D}(\gamma_0(s), s)\dot{\gamma}_0(s)\right| \int_{t_n}^{\infty} t K(t) dt + O(b_n^2)$$

from (B.16) as $\mathcal{T}^+(s) \cap \{t : |t| > t_n\} \subset (t_n, \infty)$. As $t_n \to \infty$, note that $\int_0^{t_n} tK(t)dt \to 1/2$ and Assumption A-(x) implies $K(t) t^{-(2+\eta)} \to 0$ for some $\eta > 0$ as $t \to \infty$ and hence $\int_{t_n}^{\infty} tK(t)dt \to 0$. Similarly, $\mathcal{T}^-(s) \cap \{t : |t| \leq t_n\} = [-t_n, 0]$ and thus

$$\begin{aligned} \Psi_M^-(s) &= \int_{-t_n}^{0} \int_{\gamma_0(s+b_n t)}^{\gamma_0(s)} \mathcal{D}(q, s+b_n t) K(t) dq dt + o(b_n) \\ &= -b_n \mathcal{D}(\gamma_0(s), s) \dot{\gamma}_0(s) \int_{-\infty}^{0} t K(t) dt + o(b_n), \end{aligned}$$

which yields $\mathbb{E}[\Delta M_n(s)] = \Psi_M^+(s) + \Psi_M^-(s) = b_n \mathcal{D}(\gamma_0(s), s)\dot{\gamma}_0(s) + o(b_n)$ because $\int_0^{\infty} tK(t)dt - \int_{-\infty}^{0} tK(t)dt = 1$. When $\dot{\gamma}_0(s) < 0$, we have $\mathcal{T}^+(s) \cap \{t : |t| \leq t_n\} = [-t_n, 0]$ and $\mathcal{T}^-(s) \cap \{t : |t| \leq t_n\} = [0, t_n]$. Therefore, we can symmetrically show that $\mathbb{E}[\Delta M_n(s)] = -b_n \mathcal{D}(\gamma_0(s), s)\dot{\gamma}_0(s) + o(b_n)$.

Second, we suppose $\dot{\gamma}_0(s) = 0$ and s is the local minimizer. Then, $\mathcal{T}^+(s) \cap \{t : |t| \leq t_n\} = \{t : |t| \leq t_n\}$ and hence

$$\begin{aligned} & \Psi_M^+(s) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{B.18}) \\ = \ & \int_{-t_n}^{t_n} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t) K(t) dq dt + \int_{\mathcal{T}^+(s)\cap\{t:|t|>t_n\}} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t) K(t) dq dt \\ = \ & O\left(b_n^2\right) \int_{-t_n}^{t_n} t^2 K(t) dt + o(b_n^2) \end{aligned}$$

$$= \quad O\left(b_n^2\right)$$

from (B.16), where $\int_{-t_n}^{t_n} t^2 K(t)dt \to \int_{-\infty}^{\infty} t^2 K(t)dt < \infty$ as $t_n \to \infty$ from Assumptions A-(x). Note that the second equality is because

$$\int_{\mathcal{T}^+(s)\cap\{t:|t|>t_n\}} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t)K(t)dqdt = O\left(b_n^2 \int_{|t|>t_n} t^2 K(t)dt\right) = o\left(b_n^2\right),$$

where $\mathcal{T}^+(s) \cap \{t : |t| > t_n\} \subset \{t : |t| > t_n\}$ and $\int_{|t|>t_n} t^2 K(t)dt \to 0$ as $t_n \to \infty$. However, $\mathcal{T}^-(s) \cap \{t : |t| \leq t_n\}$ becomes empty and hence

$$\Psi_M^-(s) = 0 + \int_{\mathcal{T}^-(s)\cap\{t:|t|>t_n\}} \int_{\gamma_0(s+b_n t)}^{\gamma_0(s)} \mathcal{D}(q, s+b_n t)K(t)dqdt = o(b_n^2).$$

When $\dot{\gamma}_0(s) = 0$ and s is the local maximizer, we can symmetrically show that

$$\Psi_M^+(s) = 0 + \int_{\mathcal{T}^+(s)\cap\{t:|t|>t_n\}} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}(q, s+b_n t)K(t)dqdt = o(b_n^2)$$

and

$$\Psi_M^-(s) = \int_{-t_n}^{t_n} \int_{\gamma_0(s+b_n t)}^{\gamma_0(s)} \mathcal{D}(q, s+b_n t)K(t)dqdt + o(b_n^2) = O(b_n^2).$$

By combining these results, we have $\mathbb{E}\left[\Delta M_n(s)\right] = b_n \mathcal{D}(\gamma_0(s), s)|\dot{\gamma}_0(s)| + o(b_n)$ for a given $s \in \mathcal{S}_0$, and hence

$$\sup_{s\in\mathcal{S}_0} \mathbb{E}\left[\Delta M_n(s)\right] = O(b_n)$$

since $\sup_{s\in\mathcal{S}_0} \mathcal{D}(\gamma_0(s), s)|\dot{\gamma}_0(s)| < \infty$ from Assumptions A-(vi) and (vii).

The desired result then follows if $\sup_{s\in\mathcal{S}_0} ||\Delta M_n(s) - \mathbb{E}\left[\Delta M_n(s)\right]|| = o(b_n)$ almost surely, which can be shown as Theorem 2.2 in Carbon, Francq, and Tran (2007) (see also Section 3 in Tran (1990) and Section 5 in Carbon, Tran, and Wu (1997)). Similarly as the proof of (B.12) in Lemma A.3, we let $\tau_n = (n \log n)^{1/(4+2\varphi)}$ and define

$$\Delta M_n^\tau(s) = \frac{1}{nb_n} \sum_{i\in\Lambda_n} x_i^2 \Delta_i(s_i, s)K_i(s)\mathbf{1}_{\tau_n}$$

as in (B.11), where $\Delta_i(s_i, s) = \mathbf{1}_i(\gamma_0(s_i)) - \mathbf{1}_i(\gamma_0(s))$ and $\mathbf{1}_{\tau_n} = \mathbf{1}\left\{x_i^2 \leq \tau_n\right\}$. We also let m_n be an integer such that $m_n = O(\tau_n n^{1-2\epsilon}/b_n^2)$, which diverges as $n \to \infty$, and we cover the compact $\mathcal{S}_0$ by m_n intervals centered at s_k, which are defined as $\mathcal{I}_k = \{s' : |s' - s_k| \leq C/m_n\}$ for some $C < \infty$. Then,

$$\begin{aligned}
\sup_{s\in\mathcal{S}_0} \|\Delta M_n^\tau(s) - \mathbb{E}\left[M\Delta_n^\tau(s)\right]\| \quad &\leq \quad \max_{1\leq k\leq m_n} \sup_{s\in\mathcal{I}_k} \|\Delta M_n^\tau(s) - \Delta M_n^\tau(s_k)\| \\
&\quad + \max_{1\leq k\leq m_n} \sup_{s\in\mathcal{I}_k} \|\mathbb{E}\left[\Delta M_n^\tau(s)\right] - \mathbb{E}\left[\Delta M_n^\tau(s_k)\right]\| \\
&\quad + \max_{1\leq k\leq m_n} \|\Delta M_n^\tau(s_k) - \mathbb{E}\left[\Delta M_n^\tau(s_k)\right]\| \\
&\equiv \quad \Psi_{\Delta M1} + \Psi_{\Delta M2} + \Psi_{\Delta M3}.
\end{aligned}$$

However,

$$\begin{aligned}
\|\Delta M_n^\tau(s)-\Delta M_n^\tau(s_k)\| &\leq \frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2\left|\Delta_i(s_i,s)-\Delta_i(s_i,s_k)\right| K_i(s_k)\mathbf{1}_{\tau_n} \\
&\quad +\frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2\left|\Delta_i(s_i,s)\right|\left|K_i(s)-K_i(s_k)\right|\mathbf{1}_{\tau_n} \\
&= \frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2\left|\mathbf{1}_i\left(\gamma_0\left(s_k\right)\right)-\mathbf{1}_i\left(\gamma_0\left(s\right)\right)\right| K_i(s_k)\mathbf{1}_{\tau_n} \\
&\quad +\frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i^2\left|\mathbf{1}_i\left(\gamma_0\left(s_i\right)\right)-\mathbf{1}_i\left(\gamma_0\left(s\right)\right)\right|\left|K_i(s)-K_i(s_k)\right|\mathbf{1}_{\tau_n} \\
&= O_{a.s.}\left(\frac{\tau_n}{b_n^2 m_n}\right)=O_{a.s.}\left(\frac{1}{n^{1-2\epsilon}}\right)
\end{aligned}$$

as in (B.13) and (B.14), and hence $\Psi_{\Delta M1}=\Psi_{\Delta M2}=o_{a.s.}(b_n)$ as $n^{1-2\epsilon}b_n\to\infty$. We also have $\Psi_{\Delta M3}=o_{a.s.}(b_n)$ as proved below,[12] which completes the proof. ■

Proof of $\Psi_{\Delta M3}=o_{a.s.}(b_n)$: We let

$$Z_i^\tau(s)=(nb_n)^{-1}\left\{(c_0^\top x_i)^2\Delta_i(s_i,s)K_i\left(s\right)\mathbf{1}_{\tau_n}-\mathbb{E}[(c_0^\top x_i)^2\Delta_i(s_i,s)K_i\left(s\right)\mathbf{1}_{\tau_n}]\right\}$$

and apply the blocking technique as in Carbon, Francq, and Tran (2007), p.788. For $i=(i_1,i_2)\in\Lambda_n\subset\mathbb{R}^2$, let n_1 and n_2 are the numbers of grids in two dimensions, then $|\Lambda_n|=n=n_1n_2$. Without loss of generality, we assume $n_\ell=2wr_\ell$ for $\ell=1,2$, where w and r_ℓ are constants to be specified later. For $j=(j_1,j_2)$, define

$$\begin{aligned}
U^{[1]}(j;s) &= \sum_{i_1=2j_1w+1}^{(2j_1+1)w}\ \sum_{i_2=2j_2w+1}^{(2j_2+1)w} Z_i^\tau(s), \qquad \text{(B.19)}\\
U^{[2]}(j;s) &= \sum_{i_1=2j_1w+1}^{(2j_1+1)w}\ \sum_{i_2=(2j_2+1)w+1}^{2(j_2+1)w} Z_i^\tau(s),\\
U^{[3]}(j;s) &= \sum_{i_1=(2j_1+1)w+1}^{2(j_1+1)w}\ \sum_{i_2=2j_2w+1}^{(2j_2+1)w} Z_i^\tau(s),\\
U^{[4]}(j;s) &= \sum_{i_1=(2j_1+1)w+1}^{2(j_1+1)w}\ \sum_{i_2=(2j_2+1)w+1}^{2(j_2+1)w} Z_i^\tau(s),
\end{aligned}$$

and define four blocks as

$$\mathcal{B}^{[h]}(s)=\sum_{j_1=0}^{r_1-1}\sum_{j_2=0}^{r_2-1}U^{[h]}(j;s)\quad\text{for } h=1,2,3,4,$$

[12] Unlike the Lemma A.3, We cannot directly use the results for Q_{3n}^* in Carbon, Francq, and Tran (2007) here. This is because $O((\log n/(nb_n))^{1/2})$ is not necassarily $o(b_n)$ without further restrictions.

so that $\sum_{i\in\Lambda_n} Z_i^\tau(s) = \sum_{h=1}^4 \mathcal{B}^{[h]}(s)$ and $\Psi_{\Delta M3} = \max_{1\le k\le m_n} \left|\sum_{h=1}^4 \mathcal{B}^{[h]}(s_k)\right|$. Since these four blocks have the same number of summands, it suffices to show $\max_{1\le k\le m_n} \left|\mathcal{B}^{[1]}(s_k)\right| = o_{a.s.}(b_n)$. To this end, we show that for some $\varepsilon_n = o(b_n)$,

$$\begin{aligned} \mathbb{P}\left(\max_{1\le k\le m_n} \left|\mathcal{B}^{[1]}(s_k)\right| > \varepsilon_n\right) &\le \sum_{k=1}^{m_n} \mathbb{P}\left(\left|\mathcal{B}^{[1]}(s_k)\right| > \varepsilon_n\right) \\ &\le m_n \sup_{s\in\mathcal{S}_0} \mathbb{P}\left(\left|\mathcal{B}^{[1]}(s)\right| > \varepsilon_n\right) \\ &= O(n^{-c}) \end{aligned} \tag{B.20}$$

for some $c > 1$ and hence $\sum_{n=1}^\infty \mathbb{P}\left(\max_{1\le k\le m_n} \left|\mathcal{B}^{[1]}(s_k)\right| > \varepsilon_n\right) < \infty$. Then the almost sure convergence is obtained by the Borel-Cantelli lemma.

For any $s \in \mathcal{S}_0$, $\mathcal{B}^{[1]}(s)$ is the sum of $r = r_1 r_2 = n/\left(2w^2\right)$ of $U^{[1]}(j;s)$'s. In addition, $U^{[1]}(j;s)$ is measurable with the σ-field generated by $Z_i^\tau(s)$ with i belonging to the set

$$\{i = (i_1, i_2) : 2j_\ell w + 1 \le i_\ell \le (2j_\ell + 1)w \text{ for } \ell = 1, 2\}.$$

These sets are separated by a distance of at least w. We enumerate the random variables $U^{[1]}(j;s)$ and the corresponding σ-fields with which they are measurable in an arbitrary manner, and refer to those $U^{[1]}(j;s)$'s as $U_1(s), U_2(s), \ldots, U_r(s)$. By the uniform almost sure law of large numbers in random fields (e.g., Theorem 2 in Jenish and Prucha (2009)) and the fact that $\mathbb{E}\left[K_i(s)\, b_n^{-1}\right] \le C$, we have that for any $t = 1, \ldots, r$ and $s \in \mathcal{S}_0$,

$$\begin{aligned} |U_t(s)| &\le \frac{Cw^2\tau_n}{n}\left(\frac{1}{w^2 b_n}\sum_{i_1=2j_1w+1}^{(2j_1+1)w}\sum_{i_2=2j_2w+1}^{(2j_2+1)w} |\Delta_i(s_i, s)|\, K_i(s)\right) \\ &\le \frac{Cw^2\tau_n}{n}\left(\frac{1}{w^2}\sum_{i_1=2j_1w+1}^{(2j_1+1)w}\sum_{i_2=2j_2w+1}^{(2j_2+1)w} K_i(s)\, b_n^{-1}\right) \\ &\le \frac{C'w^2\tau_n}{n} \end{aligned} \tag{B.21}$$

almost surely for some $C, C' < \infty$. From Lemma 3.6 in Carbon, Francq, and Tran (2007), we can approximate[13] $\{U_t(s)\}_{t=1}^r$ by another sequence of random variables $\{U_t^*(s)\}_{t=1}^r$ that satisfies (i) elements of $\{U_t^*(s)\}_{t=1}^r$ are independent, (ii) $U_t^*(s)$ has the same distribution as $U_t(s)$ for all $t = 1, \ldots, r$, and (iii)

$$\sum_{t=1}^r \mathbb{E}\left[|U_t^*(s) - U_t(s)|\right] \le rC'' n^{-1} w^2 \tau_n \alpha_{w^2,w^2}(w) \tag{B.22}$$

for some $C'' < \infty$. Recall that $\alpha_{w^2,w^2}(w)$ is the α-mixing coefficient defined in (8). Then, it follows that

$$\mathbb{P}\left(\mathcal{B}^{[1]}(s) > \varepsilon_n\right) \le \mathbb{P}\left(\sum_{t=1}^r |U_t^*(s) - U_t(s)| > \varepsilon_n\right) + \mathbb{P}\left(\left|\sum_{t=1}^r U_t^*(s)\right| > \varepsilon_n\right) \tag{B.23}$$

[13]This approximation is reminiscient of the Berbee's lemma (Berbee (1987)) and is based on Rio (1995), who studies the time series case. It can also be found as Lemma 4.5 in Carbon, Tran, and Wu (1997).

for any given $s \in \mathcal{S}_0$, and hence in view of (B.20) and (B.23)

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le m_n}\left|\mathcal{B}^{[1]}(s_k)\right| > \varepsilon_n\right) &\le m_n \sup_{s\in\mathcal{S}_0}\mathbb{P}\left(\sum_{t=1}^{r}|U_t^*(s)-U_t(s)| > \varepsilon_n\right) \\
&\quad + m_n \sup_{s\in\mathcal{S}_0}\mathbb{P}\left(\left|\sum_{t=1}^{r}U_t^*(s)\right| > \varepsilon_n\right) \\
&\equiv P_{U1}+P_{U2}. \qquad \text{(B.24)}
\end{aligned}$$

First, we let $\varepsilon_n = O((\log n/n)^{1/2})$. By Markov's inequality, (B.22), and Assumption A-(iii), we have

$$\begin{aligned}
P_{U1} &\le m_n \frac{rC'' n^{-1} w^2 \tau_n \alpha_{w^2,w^2}(w)}{\varepsilon_n} \\
&\le C_1 \frac{n^{1-2\epsilon}(n\log n)^{1/(4+2\varphi)}}{b_n^2}\cdot\frac{(n\log n)^{1/(4+2\varphi)}\exp(-C_1' n^{\kappa_1})}{(\log n/n)^{1/2}} \\
&\le C_1 \exp(-C_1' n^{\kappa_1})\left(\frac{\log n}{n^{1-2\epsilon}b_n}\right)^2 \frac{n^{\kappa_2}}{(\log n)^{\kappa_3}}
\end{aligned}$$

for some $\kappa_1,\kappa_2,\kappa_3 > 0$ and $C_1, C_1' < \infty$. Recall that we chose $m_n = O(\tau_n n^{1-2\epsilon}/b_n^2)$, $n = 4w^2 r$, and $\tau_n = (n\log n)^{1/(4+2\varphi)}$. Hence $P_{U1} \to 0$ as $n\to\infty$, since $\log n/(n^{1-2\epsilon}b_n) \to 0$ and the exponential term in the last inequality diminishes faster than the polynomial order.

Second, we now choose an integer w such that

$$\begin{aligned}
w &= (n/(C_w \tau_n \lambda_n))^{1/2}, \\
\lambda_n &= (n\log n)^{1/2}
\end{aligned}$$

for some large positive constant C_w. Note that, substituting λ_n and τ_n into w gives

$$w = O\left(\frac{n^{(1+\varphi)/4(2+\varphi)}}{(\log n)^{(3+\varphi)/4(2+\varphi)}}\right),$$

which diverges as $n\to\infty$ for $\varphi > 0$. Since $U_t^*(s)$ has the same distribution as $U_t(s)$, $|U_t^*(s)|$ is also uniformly bounded by $C' n^{-1}\tau_n w^2$ almost surely for all $t = 1,\dots,r$ from (B.21). Therefore, $|\lambda_n U_t^*(s)| \le 1/2$ for all t if C_w is chosen to be large enough. Using the inequality $\exp(v) \le 1+v+v^2$ for $|v|\le 1/2$, we have $\exp(\lambda_n U_t^*(s)) \le 1+\lambda_n U_t^*(s)+\lambda_n^2 U_t^*(s)^2$. Hence

$$\mathbb{E}[\exp(\lambda_n U_t^*(s))] \le 1+\lambda_n^2\mathbb{E}\left[U_t^*(s)^2\right] \le \exp\left(\lambda_n^2\mathbb{E}\left[U_t^*(s)^2\right]\right) \qquad \text{(B.25)}$$

since $\mathbb{E}[U_t^*(s)] = 0$ and $1+v \le \exp(v)$ for $v \ge 0$. Using the fact that $\mathbb{P}(X > c) \le \mathbb{E}[\exp(Xa)]/\exp(ac)$ for any random variable X and nonrandom constants a and c, and that $\{U_t^*(s)\}_{t=1}^r$ are independent, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{t=1}^{r}U_t^*(s)\right| > \varepsilon_n\right) \\
&= \mathbb{P}\left(\sum_{t=1}^{r}\lambda_n U_t^*(s) > \lambda_n\varepsilon_n\right) + \mathbb{P}\left(-\sum_{t=1}^{r}\lambda_n U_t^*(s) > \lambda_n\varepsilon_n\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mathbb{E}\left[\exp\left(\lambda_n \sum_{t=1}^{r} U_t^*(s)\right)\right] + \mathbb{E}\left[\exp\left(-\lambda_n \sum_{t=1}^{r} U_t^*(s)\right)\right]}{\exp(\lambda_n \varepsilon_n)} \\
&\le 2\exp(-\lambda_n \varepsilon_n)\exp\left(\lambda_n^2 \sum_{t=1}^{r} \mathbb{E}\left[U_t^*(s)^2\right]\right) \qquad \text{(B.26)}
\end{aligned}
$$

by (B.25). However, using the same argument as in (B.15) above, we can show that

$$
\mathbb{E}\left[U_t^*(s)^2\right] \le \sum_{\substack{1\le i_1 \le w \\ 1 \le i_2 \le w}} \mathbb{E}\left[Z_i^\tau(s)^2\right] + \sum_{\substack{i\neq j \\ 1\le i_1, i_2 \le w \\ 1 \le j_1, j_2 \le w}} Cov\left[Z_i^\tau(s), Z_j^\tau(s)\right] \le \frac{C_2 w^2}{n^2}
$$

for some $C_2 < \infty$, which does not depend on s given Assumptions A-(v) and (x). It follows that (B.26) satisfies

$$
\begin{aligned}
\sup_{s\in\mathcal{S}_0} \mathbb{P}\left(\left|\sum_{t=1}^{r} U_t^*\right| > \varepsilon_n\right) &\le 2\exp\left(-\lambda_n\varepsilon_n + \frac{C_2 \lambda_n^2 r w^2}{n^2}\right) \qquad \text{(B.27)} \\
&= 2\exp\left(-\lambda_n \varepsilon_n + C_2 \lambda_n^2 n^{-1}\right).
\end{aligned}
$$

Recall that we chose $\varepsilon_n = O((\log n/n)^{1/2})$, hence there exists $C^* > 0$ such that $\varepsilon_n = C^* \lambda_n^{-1} \log n$ and

$$
-\lambda_n \varepsilon_n + C_2 \lambda_n^2 n^{-1} = -C^* \log n + C_2 \log n = -\left(C^* - C_2\right)\log n.
$$

Therefore, in view of (B.27), we have

$$
\begin{aligned}
P_{U2} &= m_n \sup_{s\in\mathcal{S}_0} \mathbb{P}\left(\left|\sum_{t=1}^{r} U_t^*\right| > \varepsilon_n\right) \\
&\le \frac{2m_n}{n^{C^*-C_2}} = \frac{2n^{1-2\epsilon}\left(n\log n\right)^{1/(4+2\varphi)}}{n^{C^*-C_2} b_n^2} \le C_3 \left(\frac{\log n}{n^{1-2\epsilon} b_n}\right)^2 \frac{1}{(\log n)^{\kappa_4}\, n^{\kappa_5}}
\end{aligned}
$$

for some $C_3 < \infty$, $\kappa_4 = 2 - (1/\left(4+2\varphi\right)) > 2$, and $\kappa_5 = (C^* - C_2) - 3(1-2\epsilon) - (1/\left(4+2\varphi\right)) > 1$ by choosing C^* sufficiently large (e.g., $C^* > C_2 + 17/4$). Since $\log n/(n^{1-2\epsilon} b_n) \to 0$, we have $P_{U2} \le O(n^{-\kappa_5}) \to 0$ as $n \to \infty$. Therefore, the desired result follows since $\varepsilon_n = O((\log n/n)^{1/2}) = o(b_n)$ from Assumption A-(ix) and $P_{U1} + P_{U2} = O(n^{-c})$ for some $c > 1$. ■

Proof of Lemma A.6 For a given $s \in \mathcal{S}_0$, we first show (A.12). We consider the case with $\gamma(s) > \gamma_0(s)$, and the other direction can be shown symmetrically. Since $c_0^\top D(\cdot, s) c_0 f(\cdot, s)$ is continuous at $\gamma_0(s)$ and $c_0^\top D(\gamma_0(s), s) c_0 f(\gamma_0(s), s) > 0$ from Assumptions A-(vii) and (viii), there exists a sufficiently small $\overline{C}(s) > 0$ such that

$$
\underline{\ell}_D(s) = \inf_{|\gamma(s)-\gamma_0(s)|<\overline{C}(s)} c_0^\top D(\gamma(s), s) c_0 f\left(\gamma(s), s\right) > 0. \qquad \text{(B.28)}
$$

By the mean value expansion and the fact that $T_n\left(\gamma; s\right) = c_0^\top(M_n\left(\gamma(s); s\right) - M_n\left(\gamma_0(s); s\right))c_0$, we have

$$
\begin{aligned}
\mathbb{E}\left[T_n\left(\gamma; s\right)\right] &= \int\int_{\gamma_0(s)}^{\gamma(s)} \mathbb{E}\left[\left(c_0^\top x_i\right)^2 | q, s + b_n t\right] f(q, s+b_n t) K\left(t\right) dq dt \\
&= \left(\gamma(s) - \gamma_0(s)\right) c_0^\top D(\widetilde{\gamma}(s), s) c_0 f\left(\widetilde{\gamma}(s), s\right)
\end{aligned}
$$

for some $\widetilde{\gamma}(s) \in (\gamma_0(s), \gamma(s))$, which yields

$$\mathbb{E}\left[T_n\left(\gamma; s\right)\right] \geq \left(\gamma\left(s\right) - \gamma_0\left(s\right)\right) \underline{\ell}_D(s). \tag{B.29}$$

Furthermore, if we let $\Delta_i(\gamma; s) = \mathbf{1}_i\left(\gamma\left(s\right)\right) - \mathbf{1}_i\left(\gamma_0\left(s\right)\right)$ and $Z_{n,i}(s) = \left(c_0^\top x_i\right)^2 \Delta_i(\gamma; s) K_i\left(s\right) - \mathbb{E}[\left(c_0^\top x_i\right)^2 \Delta_i(\gamma; s) K_i\left(s\right)]$, using a similar argument as (B.2), we have

$$\begin{aligned} & \mathbb{E}\left[\left(T_n\left(\gamma; s\right) - \mathbb{E}\left[T_n\left(\gamma; s\right)\right]\right)^2\right] \qquad \text{(B.30)} \\ = \;& \frac{1}{n^2 b_n^2} \sum_{i \in \Lambda_n} \mathbb{E}\left[Z_{n,i}^2(s)\right] + \frac{1}{n^2 b_n^2} \sum_{i,j \in \Lambda_n, i \neq j} Cov[Z_{n,i}(s), Z_{n,j}(s)] \\ \leq \;& \frac{C_1\left(s\right)}{n b_n} \left(\gamma\left(s\right) - \gamma_0\left(s\right)\right) \end{aligned}$$

for some $C_1(s) < \infty$.

We suppose n is large enough so that $\overline{r}(s)\phi_{1n} \leq \overline{C}(s)$. Similarly as Lemma A.7 in Hansen (2000), we set γ_g for $g = 1, \ldots, \overline{g} + 1$ such that, for any $s \in \mathcal{S}_0$, $\gamma_g\left(s\right) = \gamma_0\left(s\right) + 2^{g-1}\overline{r}(s)\phi_{1n}$, where $\overline{g}$ is an integer satisfying $\gamma_{\overline{g}}\left(s\right) - \gamma_0\left(s\right) = 2^{\overline{g}-1}\overline{r}(s)\phi_{1n} \leq \overline{C}(s)$ and $\gamma_{\overline{g}+1}\left(s\right) - \gamma_0\left(s\right) > \overline{C}(s)$. Then Markov's inequality and (B.30) yield that for any fixed $\eta(s) > 0$,

$$\begin{aligned} & \mathbb{P}\left(\max_{1 \leq g \leq \overline{g}} \left| \frac{T_n\left(\gamma_g; s\right)}{\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]} - 1 \right| > \eta(s)\right) \qquad \text{(B.31)} \\ \leq \;& \mathbb{P}\left(\max_{1 \leq g \leq \overline{g}} \left| \frac{T_n\left(\gamma_g; s\right) - \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]}{\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]} \right| > \eta(s)\right) \\ \leq \;& \frac{1}{\eta^2(s)} \sum_{g=1}^{\overline{g}} \frac{\mathbb{E}\left[\left(T_n\left(\gamma_g; s\right) - \mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right)^2\right]}{\left|\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]\right|^2} \\ \leq \;& \frac{1}{\eta^2(s)} \sum_{g=1}^{\overline{g}} \frac{C_1\left(s\right)\left(n b_n\right)^{-1}\left(\gamma\left(s\right) - \gamma_0\left(s\right)\right)}{\left|\left(\gamma\left(s\right) - \gamma_0\left(s\right)\right) \underline{\ell}_D(s)\right|^2} \\ \leq \;& \frac{1}{\eta^2(s)} \sum_{g=1}^{\overline{g}} \frac{C_1\left(s\right)\left(n b_n\right)^{-1}}{2^{g-1}\overline{r}(s)\phi_{1n}\underline{\ell}_D^2(s)} \\ \leq \;& \frac{C_1\left(s\right)}{\eta^2(s)\overline{r}(s)\underline{\ell}_D^2(s)} \sum_{g=1}^{\infty} \frac{1}{2^{g-1}} \times \frac{1}{n^{2\epsilon}}, \\ \leq \;& \varepsilon(s) \end{aligned}$$

which can be arbitrarily small with large enough n. From eq. (33) of Hansen (2000), for any $\gamma\left(s\right)$ such that $\overline{r}(s)\phi_{1n} \leq \gamma\left(s\right) - \gamma_0\left(s\right) \leq \overline{C}(s)$, there exists some g^* satisfying $\gamma_{g^*}\left(s\right) - \gamma_0\left(s\right) < \gamma\left(s\right) - \gamma_0\left(s\right) < \gamma_{g^*+1}\left(s\right) - \gamma_0\left(s\right)$, and then

$$\begin{aligned} \frac{T_n\left(\gamma; s\right)}{\left|\gamma\left(s\right) - \gamma_0\left(s\right)\right|} \;&\geq\; \frac{T_n\left(\gamma_{g^*}; s\right)}{\mathbb{E}\left[T_n\left(\gamma_{g^*}; s\right)\right]} \times \frac{\mathbb{E}\left[T_n\left(\gamma_{g^*}; s\right)\right]}{\left|\gamma_{g^*+1}\left(s\right) - \gamma_0\left(s\right)\right|} \qquad \text{(B.32)} \\ &\geq\; \left\{1 - \max_{1 \leq g \leq \overline{g}} \left| \frac{T_n\left(\gamma_g; s\right)}{\mathbb{E}\left[T_n\left(\gamma_g; s\right)\right]} - 1 \right| \right\} \frac{\mathbb{E}\left[T_n\left(\gamma_{g^*}; s\right)\right]}{\left|\gamma_{g^*+1}\left(s\right) - \gamma_0\left(s\right)\right|} \end{aligned}$$

$$\geq \quad (1-\eta(s)) \frac{\left|\gamma_{g^*}(s)-\gamma_0(s)\right| \underline{\ell}_D(s)}{\left|\gamma_{g^*+1}(s)-\gamma_0(s)\right|}$$

from (B.29), where $|\gamma_{g^*}(s)-\gamma_0(s)|\underline{\ell}_D(s)/|\gamma_{g^*+1}(s)-\gamma_0(s)|$ is some finite non-zero constant by construction. Hence, in view of (B.32), we can find $C_T(s)<\infty$ such that

$$\mathbb{P}\left(\inf_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_0(s)|<\overline{C}(s)} \frac{T_n(\gamma;s)}{|\gamma(s)-\gamma_0(s)|}<C_T(s)(1-\eta(s))\right) \leq \varepsilon(s)$$

for any $\varepsilon(s)>0$. The proof for (A.13) is similar to that for (A.12) and hence omitted.

We next show (A.14). For expositional simplicity, we present the case of scalar x_i, and so is $L_n(\gamma;s)$. Similarly as (B.30), we have

$$\mathbb{E}\left[|L_n(\gamma;s)|^2\right] \leq C_2(s)\,|\gamma(s)-\gamma_0(s)| \tag{B.33}$$

for some $C_2(s)<\infty$. By defining γ_g in the same way as above, Markov's inequality and (B.33) yields that for any fixed $\eta(s)>0$,

$$\begin{aligned}
\mathbb{P}\left(\max_{1\leq g\leq \bar{g}} \frac{\left|L_n(\gamma_g;s)\right|}{\sqrt{a_n}\left(\gamma_g(s)-\gamma_0(s)\right)}>\frac{\eta(s)}{4}\right) &\leq \frac{16}{\eta^2(s)}\sum_{g=1}^{\infty}\frac{\mathbb{E}\left[L_n(\gamma_g,s)^2\right]}{a_n\left|\gamma_g(s)-\gamma_0(s)\right|^2} \qquad \text{(B.34)}\\
&\leq \frac{16}{\eta^2(s)}\sum_{g=1}^{\infty}\frac{C_2(s)}{a_n\left|\gamma_g(s)-\gamma_0(s)\right|}\\
&\leq \frac{16C_2(s)}{\eta^2(s)\overline{r}(s)}\sum_{g=1}^{\infty}\frac{1}{2^{g-1}}
\end{aligned}$$

since $a_n=\phi_{1n}^{-1}$. This probability is arbitrarily close to zero if $\overline{r}(s)$ is chosen large enough. It is worth to note that (B.34) provides the maximal (or sharp) rate of ϕ_{1n} as a_n^{-1} because we need $a_n\left|\gamma_g(s)-\gamma_0(s)\right|=O(\phi_{1n}a_n)=O(1)$ as $n\to\infty$ at most, which is also valid in (B.31).

Similarly, from Lemma A.1, we have

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq g\leq \bar{g}}\sup_{\gamma_g(s)\leq\gamma(s)\leq\gamma_{g+1}(s)}\frac{\left|L_n(\gamma;s)-L_n(\gamma_g;s)\right|}{\sqrt{a_n}\left(\gamma_g(s)-\gamma_0(s)\right)}>\frac{\eta(s)}{4}\right) \qquad \text{(B.35)}\\
\leq\;& \sum_{g=1}^{\bar{g}}\mathbb{P}\left(\sup_{\gamma_g(s)\leq\gamma(s)\leq\gamma_{g+1}(s)}\left|L_n(\gamma;s)-L_n(\gamma_g;s)\right|>\sqrt{a_n}\left(\gamma_g(s)-\gamma_0(s)\right)\frac{\eta(s)}{4}\right)\\
\leq\;& \sum_{g=1}^{\infty}\frac{C_3(s)\left|\gamma_{g+1}(s)-\gamma_g(s)\right|^2}{\eta^4(s)\,a_n^2\left|\gamma_g(s)-\gamma_0(s)\right|^4}\\
\leq\;& \frac{C_3'(s)}{\eta^4(s)\,\overline{r}(s)^2}
\end{aligned}$$

for some $C_3(s),C_3'(s)<\infty$, where $\gamma_g(s)=\gamma_0(s)+2^{g-1}\overline{r}(s)\phi_{1n}$. This probability is also arbitrarily close to zero if $\overline{r}(s)$ is chosen large enough. Since

$$\sup_{\overline{r}(s)\phi_{1n}<|\gamma(s)-\gamma_0(s)|<\overline{C}(s)}\frac{|L_n(\gamma;s)|}{\sqrt{a_n}\left(\gamma(s)-\gamma_0(s)\right)} \tag{B.36}$$

$$\leq \quad 2 \max_{1 \leq g \leq \overline{g}} \frac{\left| L_n\left(\gamma_g; s\right)\right|}{\sqrt{a_n}\left(\gamma_g\left(s\right)-\gamma_0\left(s\right)\right)} + 2 \max_{1 \leq g \leq \overline{g}} \sup_{\gamma_g(s) \leq \gamma(s) \leq \gamma_{g+1}(s)} \frac{\left| L_n\left(\gamma; s\right) - L_n\left(\gamma_g; s\right)\right|}{\sqrt{a_n}\left(\gamma_g(s)-\gamma_0\left(s\right)\right)},$$

(B.34) and (B.35) yield

$$\begin{aligned}
& \mathbb{P}\left(\sup_{\overline{r}(s)\phi_{1n} < |\gamma(s)-\gamma_0(s)| < \overline{C}(s)} \frac{|L_n\left(\gamma; s\right)|}{\sqrt{a_n}\left(\gamma(s)-\gamma_0\left(s\right)\right)} > \eta\left(s\right)\right) \\
\leq \; & \mathbb{P}\left(2 \max_{1 \leq g \leq \overline{g}} \frac{\left| L_n\left(\gamma_g; s\right)\right|}{\sqrt{a_n}\left(\gamma_g\left(s\right)-\gamma_0\left(s\right)\right)} > \frac{\eta\left(s\right)}{2}\right) \\
& + \mathbb{P}\left(2 \max_{1 \leq g \leq \overline{g}} \sup_{\gamma_g(s) \leq \gamma(s) \leq \gamma_{g+1}(s)} \frac{\left| L_n\left(\gamma; s\right) - L_n\left(\gamma_g; s\right)\right|}{\sqrt{a_n}\left(\gamma_g(s)-\gamma_0\left(s\right)\right)} > \frac{\eta\left(s\right)}{2}\right) \\
\leq \; & \varepsilon(s)
\end{aligned}$$

for any $\varepsilon(s) > 0$ if we pick $\overline{r}(s)$ sufficiently large. ■

Proof of Lemma A.7 Using the same notations in Lemma A.5, (A.4) yields

$$\begin{aligned}
& n^{\epsilon}\left(\widehat{\theta}(\widehat{\gamma}(s)) - \theta_0\right) && \text{(B.37)} \\
= \; & \left\{\frac{1}{nb_n}\widetilde{Z}(\widehat{\gamma}(s); s)^{\top}\widetilde{Z}(\widehat{\gamma}(s); s)\right\}^{-1} \\
& \times \left\{\frac{n^{\epsilon}}{nb_n}\widetilde{Z}(\widehat{\gamma}(s); s)^{\top}\widetilde{u}(s) - \frac{n^{\epsilon}}{nb_n}\widetilde{Z}(\widehat{\gamma}(s); s)^{\top}\left(\widetilde{Z}(\widehat{\gamma}(s); s) - \widetilde{Z}(\gamma_0(s_i); s)\right)\theta_0\right\} \\
\equiv \; & \Theta_{A1}^{-1}(s)\left\{\Theta_{A2}(s) - \Theta_{A3}(s)\right\}.
\end{aligned}$$

Let $M(s) \equiv \int_{-\infty}^{\infty} D(q, s) f\left(q, s\right) dq < \infty$. For the denominator $\Theta_{A1}(s)$, we have

$$\begin{aligned}
\Theta_{A1}(s) &= \begin{pmatrix} (nb_n)^{-1}\sum_{i \in \Lambda_n} x_i x_i^{\top} K_i(s) & M_n\left(\widehat{\gamma}(s); s\right) \\ M_n\left(\widehat{\gamma}(s); s\right) & M_n\left(\widehat{\gamma}(s); s\right) \end{pmatrix} && \text{(B.38)} \\
&\to_p \begin{pmatrix} M(s) & M\left(\gamma_0(s); s\right) \\ M\left(\gamma_0(s); s\right) & M\left(\gamma_0(s); s\right) \end{pmatrix},
\end{aligned}$$

where $M\left(\gamma; s\right) < \infty$ is defined in (A.2), which is continuously differentiable in γ. Note that $|M_n\left(\widehat{\gamma}(s); s\right) - M\left(\gamma_0(s); s\right)| \leq |M_n\left(\widehat{\gamma}(s); s\right) - M\left(\widehat{\gamma}(s); s\right)| + |M\left(\widehat{\gamma}(s); s\right) - M\left(\gamma_0(s); s\right)| = o_p(1)$ from Lemma A.3 and the pointwise consistency of $\widehat{\gamma}(s)$ in Lemma A.5. In addition, we have $(nb_n)^{-1}\sum_{i \in \Lambda_n} x_i x_i^{\top} K_i\left(s\right) \to_p M(s)$ from the standard kernel estimation result. Note that the probability limit of $\Theta_{A1}(s)$ is positive definite since both $M(s)$ and $M\left(\gamma_0(s); s\right)$ are positive definite and

$$M(s) - M\left(\gamma_0(s); s\right) = \int_{\gamma_0(s)}^{\infty} D(q, s) f\left(q, s\right) dq > 0$$

for any $\gamma_0(s) \in \Gamma$ from Assumption A-(viii).

For the numerator part $\Theta_{A2}(s)$, we have $\Theta_{A2}(s) = O_p(a_n^{-1/2}) = o_p(1)$ because

$$\frac{1}{\sqrt{nb_n}}\widetilde{Z}(\widehat{\gamma}(s); s)^{\top}\widetilde{u}(s) = \begin{pmatrix} (nb_n)^{-1/2}\sum_{i \in \Lambda_n} x_i u_i K_i(s) \\ J_n\left(\widehat{\gamma}(s); s\right) \end{pmatrix} = O_p\left(1\right) \qquad \text{(B.39)}$$

from Lemma A.3 and the pointwise consistency of $\widehat{\gamma}(s)$ in Lemma A.5. Note that the standard kernel estimation result gives $(nb_n)^{-1/2}\sum_{i\in\Lambda_n} x_i u_i K_i(s) = O_p(1)$. Moreover, we have

$$\Theta_{A3}(s) = \begin{pmatrix} (nb_n)^{-1}\sum_{i\in\Lambda_n} x_i x_i^\top c_0 \left\{\mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right)\right\} K_i\left(s\right) \\ (nb_n)^{-1}\sum_{i\in\Lambda_n} x_i x_i^\top c_0 \mathbf{1}_i\left(\widehat{\gamma}(s)\right)\left\{\mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right)\right\} K_i\left(s\right) \end{pmatrix} \tag{B.40}$$

and

$$\begin{aligned} & \frac{1}{nb_n}\sum_{i\in\Lambda_n} c_0^\top x_i x_i^\top \left\{\mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right)\right\} K_i\left(s\right) \qquad\qquad (\text{B.41})\\ \leq\ & \|c_0\| \left\|M_n\left(\widehat{\gamma}(s); s\right) - M_n\left(\gamma_0(s_i); s\right)\right\| \\ \leq\ & \|c_0\| \left\{\left\|M_n\left(\widehat{\gamma}(s); s\right) - M_n\left(\gamma_0(s); s\right)\right\| + O_p(b_n)\right\} \\ =\ & o_p(1), \end{aligned}$$

where the second inequality is from (A.6) and the last equality is because $M_n\left(\gamma; s\right) \to_p M\left(\gamma; s\right)$ from Lemma A.3, which is continuous in γ and $\widehat{\gamma}(s) \to_p \gamma_0(s)$ in Lemma A.5. Since

$$\begin{aligned} & \frac{1}{nb_n}\sum_{i\in\Lambda_n} x_i x_i^\top c_0 \mathbf{1}_i\left(\widehat{\gamma}(s)\right)\left\{\mathbf{1}_i\left(\widehat{\gamma}(s)\right) - \mathbf{1}_i\left(\gamma_0\left(s_i\right)\right)\right\} K_i\left(s\right) \qquad\qquad (\text{B.42})\\ \leq\ & \|c_0\| \left\|M_n\left(\widehat{\gamma}(s); s\right) - M_n\left(\gamma_0(s_i); s\right)\right\| = o_p(1) \end{aligned}$$

from (B.41), we have $\Theta_{A3}(s) = o_p(1)$ as well, which completes the proof. ■

Proof of Lemma A.8 First, for $A_n^*\left(r,s\right)$, we consider the case with $r > 0$. Let $\Delta_i(r,s) = \mathbf{1}_i\left(\gamma_0\left(s\right) + r/a_n\right) - \mathbf{1}_i\left(\gamma_0\left(s\right)\right)$ and $h_i(r,s) = \left(c_0^\top x_i\right)^2 \Delta_i(r,s) K_i\left(s\right)$. Recall that $\delta_0 = c_0 n^{-\epsilon} = c_0 (a_n/\left(nb_n\right))^{1/2}$. By change of variables and Taylor expansion, Assumptions A-(v), (viii), and (x) imply that

$$\begin{aligned} \mathbb{E}\left[A_n^*\left(r,s\right)\right] &= \frac{a_n}{nb_n}\sum_{i\in\Lambda_n} \mathbb{E}\left[h_i(r,s)\right] \qquad\qquad (\text{B.43})\\ &= a_n \iint_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} \mathbb{E}\left[\left(c_0^\top x_i\right)^2 | q, s+b_n t\right] K\left(t\right) f\left(q, s+b_n t\right) dq dt \\ &= r c_0^\top D\left(\gamma_0\left(s\right), s\right) c_0 f\left(\gamma_0\left(s\right), s\right) + O\left(\frac{1}{a_n} + b_n^2\right), \end{aligned}$$

where the third equality holds under Assumption A-(vi). Next, we have

$$\begin{aligned} Var\left[A_n^*\left(r,s\right)\right] &= \frac{a_n^2}{n^2 b_n^2} Var\left[\sum_{i\in\Lambda_n} h_i(r,s)\right] \qquad\qquad (\text{B.44})\\ &= \frac{a_n^2}{nb_n^2} Var\left[h_i(r,s)\right] + \frac{a_n^2}{n^2 b_n^2}\sum_{\substack{i,j\in\Lambda_n \\ i\neq j}} Cov\left[h_i(r,s), h_j(r,s)\right] \\ &\equiv \Psi_{A1}(r,s) + \Psi_{A2}(r,s). \end{aligned}$$

Taylor expansion and Assumptions A-(vii), (viii), and (x) lead to

$$\Psi_{A1}(r,s) = \frac{a_n}{nb_n}\left(\frac{a_n}{b_n}\mathbb{E}\left[\left(c_0^\top x_i\right)^4 \Delta_i(r,s) K_i^2\left(s\right)\right]\right) - \frac{1}{n}\left(\frac{a_n}{b_n}\mathbb{E}\left[\left(c_0^\top x_i\right)^2 \Delta_i(r,s) K_i\left(s\right)\right]\right)^2$$

$$= O\left(\frac{a_n}{nb_n}+\frac{1}{n}\right)=O\left(n^{-2\epsilon}+\frac{1}{n}\right)$$

since $\Delta_i(r,s)^2=\Delta_i(r,s)$ for $r>0$, where each moment term is bounded as in (B.43). For Ψ_{A2}, we define a sequence of integers $\kappa_n=O\left(n^{\ell}\right)$ for some $\ell>0$ such that $\kappa_n\to\infty$ and $\kappa_n^2/n\to 0$, and decompose

$$\begin{aligned}\Psi_{A2}(r,s) &= \frac{a_n^2}{n^2b_n^2}\sum_{\substack{i,j\in\Lambda_n\\ 0<\lambda(i,j)\le\kappa_n}} Cov\left[h_i(r,s),h_j(r,s)\right]+\frac{a_n^2}{n^2b_n^2}\sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)>\kappa_n}} Cov\left[h_i(r,s),h_j(r,s)\right]\\ &= \Psi'_{A2}(r,s)+\Psi''_{A2}(r,s).\end{aligned}$$

Then, since

$$\begin{aligned}& Cov\left[\frac{a_n}{b_n}h_i(r,s),\frac{a_n}{b_n}h_j(r,s)\right]\\ \le\ & r^2\mathbb{E}\left[\left(c_0^\top x_i\right)^2\left(c_0^\top x_j\right)^2|(q_i,q_j,s_i,s_j)=\left(\gamma_0\left(s\right),\gamma_0\left(s\right),s,s\right)\right]f\left(\gamma_0\left(s\right),\gamma_0\left(s\right),s,s\right)+o\left(1\right)\end{aligned}$$

using a similar argument as in (B.4) and (B.43), similarly as the proof of $\Psi_{14,3}^{[1]}(s)$ in Lemma A.1, we have

$$\Psi'_{A2}(r,s)\le Cr^2\kappa_n^2/n=o\left(1\right)$$

for some $C<\infty$. Furthermore, by the covariance inequality (A.1) and Assumption A-(iii), we have

$$\begin{aligned}\left|\Psi''_{A2}(r,s)\right| &\le \frac{C'}{n^2}\left(\frac{a_n}{b_n}\right)^{\frac{2+2\varphi}{2+\varphi}}\sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)>\kappa_n}}\alpha_{1,1}\left(\lambda\left(i,j\right)\right)^{\varphi/(2+\varphi)}\mathbb{E}\left[\frac{a_n}{b_n}\left|h_i(r,s)\right|^{2+\varphi}\right]^{2/(2+\varphi)}\\ &\le \frac{C''}{n}\left(\frac{a_n}{b_n}\right)^{\frac{2+2\varphi}{2+\varphi}}\sum_{i\in\Lambda_n}\sum_{m=\kappa_n+1}^{n-1}\sum_{\substack{j\in\Lambda_n\\ \lambda(i,j)\in[m,m+1)}}\alpha_{1,1}\left(m\right)^{\varphi/(2+\varphi)}\\ &\le \frac{C''}{n}\left(\frac{a_n}{b_n}\right)^{\frac{2+2\varphi}{2+\varphi}}\sum_{m=\kappa_n+1}^{\infty}m\exp\left(-m\varphi/(2+\varphi)\right)\\ &= O\left(n^{((1-2\epsilon)(2+2\varphi)/(2+\varphi))-1}\kappa_n\exp(-\kappa_n\varphi/(2+\varphi))\right)\\ &= o\left(1\right),\end{aligned}$$

similarly as the proof of $\Psi_{14,3}^{[2]}(s)$ in Lemma A.1, because $\mathbb{E}[(a_n/b_n)\left|h_i(r,s)\right|^{2+\varphi}]$ is bounded as in (B.43) and we set κ_n such that $\kappa_n=O(n^{\ell})$ for $\ell>0$. Hence, the pointwise convergence of $A_n^*\left(r,s\right)$ is obtained. Furthermore, since $A_n^*(r,s)$ is monotonically increasing in r and the limit function $rc_0^\top D\left(\gamma_0\left(s\right),s\right)c_0 f\left(\gamma_0\left(s\right),s\right)$ is continuous in r, the convergence holds uniformly on any compact set. Symmetrically, we can show that $\mathbb{E}\left[A_n^*\left(r,s\right)\right]=-rc_0^\top D\left(\gamma_0\left(s\right),s\right)c_0 f\left(\gamma_0\left(s\right),s\right)+O\left(a_n^{-1}+b_n^2\right)$ when $r<0$. The uniform convergence also holds in this case using the same argument as above, which completes the proof for $A_n^*\left(r,s\right)$.

For $B_n^*\left(r,s\right)$, Assumption ID-(i) leads to $\mathbb{E}\left[B_n^*\left(r,s\right)\right]=0$. Let $\widetilde{h}_i(r,s)=c_0^\top x_i u_i\Delta_i(r,s)K_i\left(s\right)$

and write

$$
\begin{aligned}
Var[B_n^*(r,s)] &= \frac{a_n}{b_n} Var[\widetilde{h}_i(r,s)] + \frac{a_n}{nb_n} \sum_{\substack{i,j\in\Lambda_n\\ i\neq j}} Cov[\widetilde{h}_i(r,s), \widetilde{h}_j(r,s)] \\
&\equiv \Psi_{B1}(r,s) + \Psi_{B2}(r,s).
\end{aligned}
$$

As in (B.43), we have

$$
\Psi_{B1}(r,s) = |r|\, c_0^\top V(\gamma_0(s), s)\, c_0 f(\gamma_0(s), s) \int K^2(v)dv + O\left(\frac{1}{a_n} + b_n^2\right),
$$

which is nonsingular for $|r| > 0$ from Assumption A-(viii). For $\Psi_{B2}(r,s)$, we define a sequence of integers $\kappa_n' = O(n^{\ell'})$ for some $\ell' > 0$ such that $\kappa_n' \to \infty$ and $(\kappa_n')^2/n^{1-2\epsilon} \to 0$, and decompose

$$
\begin{aligned}
\Psi_{B2}(r,s) &= \frac{a_n}{nb_n} \sum_{\substack{i,j\in\Lambda_n\\ 0<\lambda(i,j)\leq\kappa_n'}} Cov[\widetilde{h}_i(r,s), \widetilde{h}_j(r,s)] + \frac{a_n}{nb_n} \sum_{\substack{i,j\in\Lambda_n\\ \lambda(i,j)>\kappa_n'}} Cov[\widetilde{h}_i(r,s), \widetilde{h}_j(r,s)] \\
&\equiv \Psi_{B2}'(r,s) + \Psi_{B2}''(r,s).
\end{aligned}
$$

Then similarly as Ψ_{A2}' and Ψ_{A2}'' above, we have

$$
\begin{aligned}
\left|\Psi_{B2}'(r,s)\right| &\leq Cr^2(\kappa_n')^2 \times \frac{b_n}{a_n} = O\left(\frac{(\kappa_n')^2}{n^{1-2\epsilon}}\right) = o(1), \\
\left|\Psi_{B2}''(r,s)\right| &\leq C'\left(\frac{a_n}{b_n}\right)^{\varphi/(2+\varphi)} \sum_{m=\kappa_n+1}^{\infty} m\exp\left(-m\varphi/(2+\varphi)\right) \\
&= C' n^{(1-2\epsilon)\varphi/(2+\varphi)} \kappa_n' \exp(-\kappa_n'\varphi/(2+\varphi)) = o(1)
\end{aligned}
$$

for some $C, C' < \infty$. By combining these results, we have

$$
Var[B_n^*(r,s)] = |r|\, c_0^\top V(\gamma_0(s), s)\, c_0 f(\gamma_0(s), s)\, \kappa_2 + o(1)
$$

with $\kappa_2 = \int K^2(v)dv$, and by the CLT for stationary and mixing random field (e.g., Bolthausen (1982) and Jenish and Prucha (2009)), we have

$$
B_n^*(r,s) \Rightarrow W(r)\sqrt{c_0^\top V(\gamma_0(s), s)\, c_0 f(\gamma_0(s), s)\, \kappa_2}
$$

as $n \to \infty$, where $W(r)$ is the two-sided Brownian Motion defined in (10).

This pointwise convergence in r can be extended to any finite-dimensional convergence in r by the fact that $Cov[B_n^*(r_1,s), B_n^*(r_2,s)] = Var[B_n^*(r_1,s)] + o(1)$ for any $r_1 < r_2$, which is because $(\mathbf{1}_i(\gamma_0 + r_2/a_n) - \mathbf{1}_i(\gamma_0 + r_1/a_n))\, \mathbf{1}_i(\gamma_0 + r_1/a_n) = 0$. The tightness follows from a similar argument as $J_n(\gamma; s)$ in Lemma A.1 and the desired result follows by Theorem 15.5 in Billingsley (1968). ■

Proof of Lemma A.9 For the first result, using the same notations in Lemma A.5, we write

$$
\begin{aligned}
&\sqrt{nb_n}\left(\widehat{\theta}(\widehat{\gamma}(s)) - \theta_0\right) \\
&= \left\{\frac{1}{nb_n}\widetilde{Z}(\widehat{\gamma}(s); s)^\top \widetilde{Z}(\widehat{\gamma}(s); s)\right\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& \times \left\{ \frac{1}{\sqrt{nb_n}} \widetilde{Z}(\widehat{\gamma}(s);s)^\top \widetilde{u}(s) - \frac{1}{\sqrt{nb_n}} \widetilde{Z}(\widehat{\gamma}(s);s)^\top \left(\widetilde{Z}(\widehat{\gamma}(s);s) - \widetilde{Z}(\gamma_0(s_i);s) \right) \theta_0 \right\} \\
\equiv \;\; & \Theta_{B1}^{-1}(s) \left\{ \Theta_{B2}(s) - \Theta_{B3}(s) \right\}
\end{aligned}
$$

similarly as (B.37). For the denominator, since $\Theta_{B1}(s) = \Theta_{A1}(s)$ in (B.37), then $\Theta_{B1}^{-1}(s) = O_p(1)$ from (B.38). For the numerator, we first have $\Theta_{B2}(s) = O_p(1)$ from (B.39). For $\Theta_{B3}(s)$, similarly as (B.40),

$$
\Theta_{B3}(s) = \begin{pmatrix} a_n^{-1/2} \sum_{i \in \Lambda_n} n^{-\epsilon} x_i x_i^\top \delta_0 \left\{ \mathbf{1}_i \left(\widehat{\gamma}(s) \right) - \mathbf{1}_i \left(\gamma_0 \left(s_i \right) \right) \right\} K_i \left(s \right) \\ a_n^{-1/2} \sum_{i \in \Lambda_n} n^{-\epsilon} x_i x_i^\top \delta_0 \mathbf{1}_i \left(\widehat{\gamma}(s) \right) \left\{ \mathbf{1}_i \left(\widehat{\gamma}(s) \right) - \mathbf{1}_i \left(\gamma_0 \left(s_i \right) \right) \right\} K_i \left(s \right) \end{pmatrix}.
$$

However, since $\widehat{\gamma}(s) = \gamma_0(s) + r(s)\phi_{1n}$ for some $r(s)$ bounded in probability from Theorem 2, similarly as (B.43), we have

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{i \in \Lambda_n} n^{-\epsilon} \delta_0^\top x_i x_i^\top \left\{ \mathbf{1}_i \left(\widehat{\gamma}(s) \right) - \mathbf{1}_i \left(\gamma_0 \left(s_i \right) \right) \right\} K_i \left(s \right) \right] \\
\leq \;\; & a_n \left| \iint_{\min\{\gamma_0(s+b_n t), \gamma_0(s)+r(s)\phi_{1n}\}}^{\max\{\gamma_0(s+b_n t), \gamma_0(s)+r(s)\phi_{1n}\}} \mathbb{E}\left[x_i x_i^\top c_0 | q, s+b_n t \right] K\left(t\right) f\left(q, s+b_n t\right) dq dt \right| \\
\leq \;\; & a_n \left| \iint_{\min\{\gamma_0(s)+r(s)\phi_{1n}, \gamma_0(s)\}}^{\max\{\gamma_0(s)+r(s)\phi_{1n}, \gamma_0(s)\}} \mathbb{E}\left[x_i x_i^\top c_0 | q, s+b_n t \right] K\left(t\right) f\left(q, s+b_n t\right) dq dt \right| \\
& + a_n \left| \iint_{\min\{\gamma_0(s+b_n t), \gamma_0(s)\}}^{\max\{\gamma_0(s+b_n t), \gamma_0(s)\}} \mathbb{E}\left[x_i x_i^\top c_0 | q, s+b_n t \right] K\left(t\right) f\left(q, s+b_n t\right) dq dt \right| \\
= \;\; & a_n \phi_{1n} \left| r(s) \right| \left| D\left(\gamma_0\left(s\right), s \right) c_0 \right| f\left(\gamma_0\left(s\right), s \right) + O(a_n b_n) \\
= \;\; & O(1)
\end{aligned}
$$

as $a_n \phi_{1n} = 1$ and $a_n b_n = n^{1-2\epsilon} b_n^2 \to \varrho < \infty$. We also have

$$
Var\left[\sum_{i \in \Lambda_n} n^{-\epsilon} x_i x_i^\top \delta_0 \left\{ \mathbf{1}_i \left(\widehat{\gamma}(s) \right) - \mathbf{1}_i \left(\gamma_0 \left(s_i \right) \right) \right\} K_i \left(s \right) \right] = O(n^{-2\epsilon}) = o(1),
$$

similarly as (B.44). Therefore, from the same reason as (B.42), we have $\Theta_{B3}(s) = O_p(a_n^{-1/2}) = o_p(1)$, which completes the proof.

For the second result, given the same derivation for $\Theta_{B1}^{-1}(s)$ and $\Theta_{B3}(s)$ above, it suffices to show that

$$
\frac{1}{\sqrt{nb_n}} \widetilde{Z}(\widehat{\gamma}(s);s)^\top \widetilde{u}(s) - \frac{1}{\sqrt{nb_n}} \widetilde{Z}(\gamma_0(s);s)^\top \widetilde{u}(s) = o_p(1),
$$

which is implied by Lemma A.1. ■

Proof of Lemma A.10 First, we consider the case with $r > 0$. For a fixed $s \in \mathcal{S}_0$, we have

$$
\begin{aligned}
& \left\{ \mathbf{1}[q \leq \gamma_0\left(s\right) + r/a_n] - \mathbf{1}[q \leq \gamma_0\left(s\right)] \right\} \left\{ \mathbf{1}[q \leq \gamma_0\left(s + b_n t\right)] - \mathbf{1}[q \leq \gamma_0\left(s\right)] \right\} \\
= \;\; & \begin{cases} \mathbf{1}\left[\gamma_0\left(s\right) < q \leq \gamma_0\left(s + b_n t\right) \right] & \text{if } \gamma_0\left(s + b_n t\right) \leq \gamma_0\left(s\right) + r/a_n, \\ \mathbf{1}\left[\gamma_0\left(s\right) < q \leq \gamma_0\left(s\right) + r/a_n \right] & \text{otherwise.} \end{cases}
\end{aligned}
$$

Denote $\mathcal{D}_{c_0}(q,s) = c_0^\top D(q,s)c_0 f(q,s)$. Then

$$\begin{aligned}
& \mathbb{E}\left[B_n^{**}(r,s)\right] \\
= \;& a_n \iint c_0^\top D(q, s+b_n t) c_0 \left\{\mathbf{1}[q \le \gamma_0(s) + r/a_n] - \mathbf{1}[q \le \gamma_0(s)]\right\} \\
& \qquad \times \left\{\mathbf{1}[q \le \gamma_0(s+b_n t)] - \mathbf{1}[q \le \gamma_0(s)]\right\} K(t) f(q, s+b_n t)\, dqdt \\
= \;& a_n \int_{\mathcal{T}_1^*(r;s)} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dqdt \\
& + a_n \int_{\mathcal{T}_2^*(r;s)} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dqdt \\
\equiv \;& B_{n1}^{**}(r,s) + B_{n2}^{**}(r,s),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_1^*(r;s) &= \{t : \gamma_0(s) < \gamma_0(s+b_n t)\} \cap \{t : \gamma_0(s+b_n t) \le \gamma_0(s) + r/a_n\}, \\
\mathcal{T}_2^*(r;s) &= \{t : \gamma_0(s) < \gamma_0(s+b_n t)\} \cap \{t : \gamma_0(s) + r/a_n < \gamma_0(s+b_n t)\}.
\end{aligned}$$

Note that $\gamma_0(s) < \gamma_0(s) + r/a_n$ always holds for $r > 0$. Similarly as in the proof of Lemma A.4, we let a positive sequence $t_n \to \infty$ such that $t_n b_n \to 0$ as $n \to \infty$. Since $\int_{t_n}^\infty tK(t)dt \to 0$ by Assumption A-(x) with $t_n \to \infty$, both $\mathcal{T}_1^*(r;s) \cap \{t : |t| > t_n\}$ and $\mathcal{T}_2^*(r;s) \cap \{t : |t| > t_n\}$ becomes negligible as $t_n \to \infty$ using the same argument in (B.17). It follows that

$$\begin{aligned}
B_{n1}^{**}(r,s) &= a_n \int_{\mathcal{T}_1(r;s)} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dqdt + o(a_n b_n), \\
B_{n2}^{**}(r,s) &= a_n \int_{\mathcal{T}_2(r;s)} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dqdt + o(a_n b_n),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_1(r;s) &= \mathcal{T}_1^*(r;s) \cap \{t : |t| \le t_n\}, \\
\mathcal{T}_2(r;s) &= \mathcal{T}_2^*(r;s) \cap \{t : |t| \le t_n\}.
\end{aligned}$$

Recall that $a_n b_n = n^{1-2\epsilon} b_n^2 \to \varrho < \infty$ and hence $o(a_n b_n) = o(1)$. We consider three cases of $\dot\gamma_0(s) > 0$, $\dot\gamma_0(s) < 0$, and $\dot\gamma_0(s) = 0$ separately.

First, we suppose $\dot\gamma_0(s) > 0$. For any fixed $\varepsilon > 0$, it holds $t_n b_n \le \varepsilon$ if n is sufficiently large. Therefore, for both $\mathcal{T}_1(r;s)$ and $\mathcal{T}_2(r;s)$, $\gamma_0(s) < \gamma_0(s+b_n t)$ requires that $t > 0$ for sufficiently large n. Furthermore, $\gamma_0(s+b_n t) < \gamma_0(s) + r/a_n$ implies that $t < r/(a_n b_n \dot\gamma_0(\widetilde{s}))$ for some $\widetilde{s} \in [s, s+b_n t]$, where $0 < r/(a_n b_n \dot\gamma_0(\widetilde{s})) < \infty$. Therefore, $\mathcal{T}_1(r;s) = \{t : t > 0$ and $t < r/(a_n b_n \dot\gamma_0(\widetilde{s}))\}$ for sufficiently large n. It follows that, by Taylor expansion,

$$\begin{aligned}
B_{n1}^{**}(r,s) &= a_n \int_0^{r/(a_n b_n \dot\gamma_0(\widetilde{s}))} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dqdt \\
&= a_n b_n \mathcal{D}_{c_0}(\gamma_0(s), s)\dot\gamma_0(s) \int_0^{r/(a_n b_n \dot\gamma_0(\widetilde{s}))} tK(t)\, dt + a_n b_n O(b_n) \\
&= \varrho \mathcal{D}_{c_0}(\gamma_0(s), s)\dot\gamma_0(s) \mathcal{K}_1(r, \varrho; s) + o(1)
\end{aligned}$$

for sufficiently large n, since $a_n b_n = n^{1-2\epsilon} b_n^2 \to \varrho < \infty$ and $\widetilde{s} \to s$ as $n \to \infty$. Similarly, since $\gamma_0(s) + r/a_n < \gamma_0(s + b_n t)$ implies $t > r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))$ for some $\widetilde{s} \in [s, s + b_n t]$, we have $\mathcal{T}_2(r; s) = \{t : t > 0 \text{ and } t > r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))\}$. Hence,

$$\begin{aligned}
B^{**}_{n2}(r,s) &= a_n \int_{r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))}^{t_n} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dq dt \\
&= r\mathcal{D}_{c_0}(\gamma_0(s), s) \int_{r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))}^{t_n} K(t)\, dt + O(b_n) \\
&= r\mathcal{D}_{c_0}(\gamma_0(s), s) \left\{ \frac{1}{2} - \mathcal{K}_0(r, \varrho; s) \right\} + o(1)
\end{aligned}$$

for sufficiently large n. Recall that $|\mathcal{K}_0(r, \varrho; s)| \leq 1/2$ and $|\mathcal{K}_1(r, \varrho; s)| \leq 1/2$.

When $\dot{\gamma}_0(s) < 0$, $-\infty < r/(a_n b_n \dot{\gamma}_0(s)) < 0$ and we can similarly derive

$$\begin{aligned}
B^{**}_{n1}(r,s) &= a_n \int_{r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))}^{0} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dq dt \\
&= -\varrho \mathcal{D}_{c_0}(\gamma_0(s), s) \dot{\gamma}_0(s) \mathcal{K}_1(r, \varrho; s) + o(1), \\
B^{**}_{n2}(r,s) &= a_n \int_{-t_n}^{r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dq dt \\
&= r\mathcal{D}_{c_0}(\gamma_0(s), s) \left\{ \frac{1}{2} - \mathcal{K}_0(r, \varrho; s) \right\} + o(1).
\end{aligned}$$

When $\dot{\gamma}_0(s) = 0$, it suffices to consider $\gamma_0(s)$ as the local minimum, so that $\dot{\gamma}_0(t) \leq 0$ for $t \in [s - \varepsilon, s]$ and $\dot{\gamma}_0(t) \geq 0$ for $t \in [s, s + \varepsilon]$ for some small ε. In this case, based on the same argument as (B.18),

$$\begin{aligned}
\mathcal{T}_1(r; s) &= \{t : \gamma_0(s + b_n t) \leq \gamma_0(s) + r/a_n\} \cap \{t : |t| \leq t_n\}, \\
\mathcal{T}_2(r; s) &= \{t : \gamma_0(s) + r/a_n < \gamma_0(s + b_n t)\} \cap \{t : |t| \leq t_n\}.
\end{aligned}$$

Therefore, for sufficiently large n,

$$\begin{aligned}
B^{**}_{n1}(r,s) &= a_n \int_{0}^{t_n} \int_{\gamma_0(s)}^{\gamma_0(s+b_n t)} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dq dt \\
&= -\varrho \mathcal{D}_{c_0}(\gamma_0(s), s) \dot{\gamma}_0(s) \int_0^\infty t K(t)\, dt + o(1) = o(1), \\
B^{**}_{n2}(r,s) &= a_n \int_{-t_n}^{r/(a_n b_n \dot{\gamma}_0(\widetilde{s}))} \int_{\gamma_0(s)}^{\gamma_0(s)+r/a_n} \mathcal{D}_{c_0}(q, s+b_n t) K(t)\, dq dt \\
&= r\mathcal{D}_{c_0}(\gamma_0(s), s) \left\{ \frac{1}{2} - \mathcal{K}_0(r, \varrho; s) \right\} + o(1) = o(1)
\end{aligned}$$

since $\mathcal{K}_0(r, \varrho; s) = 1/2$ when $\dot{\gamma}_0(s) = 0$.

By combining all three cases and the symmetric argument for $r < 0$, we have

$$\mathbb{E}\left[B^{**}_n(r,s)\right] = |r|\, \mathcal{D}_{c_0}(\gamma_0(s), s) \left\{ \frac{1}{2} - \mathcal{K}_0(r, \varrho; s) \right\} + \varrho \mathcal{D}_{c_0}(\gamma_0(s), s)\, |\dot{\gamma}_0(s)|\, \mathcal{K}_1(r, \varrho; s) + o(1).$$

Furthermore, we have $|B^{**}_n(r,s)| \leq \sum_{i \in \Lambda_n} (\delta_0^\top x_i)^2 \left| \mathbf{1}_i(\gamma_0(s) + r/a_n) - \mathbf{1}_i(\gamma_0(s)) \right| K_i(s)$ and hence $Var\left[B^{**}_n(r,s)\right] = O(n^{-2\epsilon}) = o(1)$ from (B.44), which establishes the pointwise convergence for each r. The tightness follows from a similar argument as in Lemma A.1 and the desired result follows by Theorem 15.5 in Billingsley (1968). ■

Proof of Lemma A.11 Define $W_\mu(r) = W(r) + \mu(r)$, $\tau^+ = \arg\max_{r\in\mathbb{R}^+} W_\mu(r)$, and $\tau^- = \arg\max_{r\in\mathbb{R}^-} W_\mu(r)$. The process $W_\mu(\cdot)$ is a Gaussian process, and hence Lemma 2.6 of Kim and Pollard (1990) implies that τ^+ and τ^- are unique almost surely. Recall that we define $W(r) = W_1(-r)1[r < 0] + W_2(r)1[r > 0]$, where $W_1(\cdot)$ and $W_2(\cdot)$ are two independent standard Wiener processes defined on $\mathbb{R}^+$. We claim that

$$\mathbb{E}[\tau^+] = -\mathbb{E}[\tau^-] < \infty, \tag{B.45}$$

which gives the desired result.

The equality in (B.45) follows directly from the symmetry (i.e., $\mathbb{P}(\tau^+ \leq t) = \mathbb{P}(\tau^- \geq -t)$ for any $t > 0$) and the fact that W_1 is independent of W_2. Now, we focus on $r > 0$ and show that $\mathbb{E}[\tau^+] < \infty$. First, for any $r > 0$,

$$\mathbb{P}\left(W_\mu(r) \geq 0\right) = \mathbb{P}\left(W_2(r) \geq -\mu(r)\right) = \mathbb{P}\left(\frac{W_2(r)}{\sqrt{r}} \geq -\frac{\mu(r)}{\sqrt{r}}\right) = 1 - \Phi\left(-\frac{\mu(r)}{\sqrt{r}}\right),$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. Since the sample path of $W_\mu(\cdot)$ is continuous, for some $\underline{r} > 0$, we then have

$$\begin{aligned}
\mathbb{E}[\tau^+] &= \int_0^\infty \left\{1 - \mathbb{P}\left(\tau^+ \leq r\right)\right\} dr \\
&= \int_0^{\underline{r}} \mathbb{P}\left(\tau^+ > r\right) dr + \int_{\underline{r}}^\infty \mathbb{P}\left(\tau^+ > r\right) dr \\
&\leq C_1 + \int_{\underline{r}}^\infty \mathbb{P}\left(W_\mu(\tau^+) \geq 0 \textit{ and } \tau^+ > r\right) dr \\
&\leq C_1 + \int_{\underline{r}}^\infty \mathbb{P}\left(W_\mu(r) \geq 0\right) dr \\
&= C_1 + \int_{\underline{r}}^\infty \left(1 - \Phi\left(-\frac{\mu(r)}{\sqrt{r}}\right)\right) dr
\end{aligned} \tag{B.46}$$

for some $C_1 < \infty$, where the first inequality is because $W_\mu(\tau^+) = \max_{r\in\mathbb{R}^+} W_\mu(r) \geq 0$ given $W_\mu(0) = 0$, and the second inequality is because $\mathbb{P}\left(W_\mu(r) \geq 0\right)$ is monotonically decreasing to zero on $[\underline{r}, \infty)$ by assumption. The second term in (B.46) can be bounded as follows. Using the change of variables $t = r^\varepsilon$, integral by parts, and the condition that $r^{-(1/2+\varepsilon)}\mu(r)$ monotonically decreases to $-\infty$ on $[\underline{r}, \infty)$ for some $\varepsilon > 0$, we have

$$\begin{aligned}
\int_{\underline{r}}^\infty \left(1 - \Phi\left(-\frac{\mu(r)}{\sqrt{r}}\right)\right) dr &\leq C_2 \int_{\underline{r}}^\infty \left(1 - \Phi\left(r^\varepsilon\right)\right) dr \\
&= C_3 \int_{\underline{r}^{1/\varepsilon}}^\infty \left(1 - \Phi\left(t\right)\right) dt^{1/\varepsilon} \\
&= C_4 + C_5 \int_{\underline{r}^{1/\varepsilon}}^\infty t^{1/\varepsilon}\phi(t) dt < \infty
\end{aligned}$$

for some $C_j < \infty$ for $j = 2, 3, 4, 5$, where $\phi(\cdot)$ denotes the standard normal density function and we use $\lim_{t\to\infty} t^{1/\varepsilon}\left(1 - \Phi\left(t\right)\right) = 0$. The same result can be obtained for $r < 0$ symmetrically, which completes the proof. ■

Proof of Lemma A.12 For given (ϱ, s), we simply denote $\mu(r) = \mu(r, \varrho; s)$. Then, for the kernel functions satisfying Assumption A-(x), it is readily verified that $\mu(0) = 0$, $\mu(r)$ is continuous in r, and $\mu(r)$ is symmetric about zero. To check the monotonically decreasing condition, for $r > 0$, we write

$$\mu(r) = -r \int_0^{rC_1} K(t)dt + C_2 \int_0^{rC_1} tK(t)dt,$$

where C_1 and C_2 are some positive constants depending on $(\varrho, |\dot{\gamma}_0(s)|, \xi(s))$ from (A.22). We consider the two possible cases.

First, if $K(\cdot)$ has a bounded support, say $[-\underline{r}, \underline{r}]$ for some $0 < \underline{r} < \infty$, then $\mu(r) = -rC_3 + C_4$ for $r > \underline{r}$ and some $0 < C_3, C_4 < \infty$. Thus, $\mu(r)r^{-(1/2+\varepsilon)}$ is monotonically decreasing to $-\infty$ on $[\underline{r}, \infty)$ for any $\varepsilon \in (0, 1/2)$.

Second, if $K(\cdot)$ has an unbounded support,

$$\mu(r)r^{-((1/2)+\varepsilon)} = -r^{1/2-\varepsilon} \int_0^{rC_1} K(t)dt + r^{-(1/2+\varepsilon)} C_2 \int_0^{rC_1} tK(t)dt,$$

which goes to $-\infty$ as $r \to \infty$ since $\int_0^{rC_1} tK(t)dt \le \int_0^{\infty} tK(t)dt < \infty$ and $\int_0^{rC_1} K(t)dt > 0$. We can verify the monotonicity since

$$\begin{aligned}
\frac{\partial}{\partial r}\left\{\mu(r)r^{-((1/2)+\varepsilon)}\right\} &= -\left(\frac{1}{2} - \varepsilon\right) r^{-(1/2+\varepsilon)} \int_0^{rC_1} K(t)dt - r^{1/2-\varepsilon} C_1 K(C_1 r) \\
&\quad - \left(\frac{1}{2} + \varepsilon\right) r^{-(3/2+\varepsilon)} C_2 \int_0^{rC_1} tK(t)dt + r^{1/2-\varepsilon} C_1^2 C_2 K(C_1 r) \\
&= -r^{-(1/2+\varepsilon)} \left\{ \left(\frac{1}{2} - \varepsilon\right) \int_0^{rC_1} K(t)dt + rK(C_1 r)\left(C_1 - C_1^2 C_2\right) \right\} \\
&\quad - \left(\frac{1}{2} + \varepsilon\right) r^{-3/2-\varepsilon} C_2 \int_0^{rC_1} tK(t)dt
\end{aligned}$$

by the Leibniz integral rule. For $r > \underline{r}$ for some large enough $\underline{r}$ and $\varepsilon \in (0, 1/2)$, this derivative is strictly negative because $(1/2 - \varepsilon) \int_0^{rC_1} K(t)dt > 0$ and $\lim_{r\to\infty} rK(r) = 0$, which proves $\mu(r)r^{-((1/2)+\varepsilon)}$ is monotonically decreasing on $[\underline{r}, \infty)$. The case with $r < 0$ follows symmetrically. ■

To prove Lemma A.13, we first present the following two lemmas.

Lemma B.3 *There exist constants C^* and $\bar{C}^*$ such that for any $\gamma(\cdot) \in \mathcal{G}_n(\mathcal{S}_0; \Gamma)$*

$$\begin{aligned}
\sup_{s \in \mathcal{S}_0} |T_n(\gamma; s) - \mathbb{E}[T_n(\gamma; s)]| &\le C^* \left(\sup_{s \in \mathcal{S}_0} |\gamma(s) - \gamma_0(s)| \frac{\log n}{nb_n} \right)^{1/2} \\
\sup_{s \in \mathcal{S}_0} \left|\overline{T}_n(\gamma; s) - \mathbb{E}\left[\overline{T}_n(\gamma; s)\right]\right| &\le \bar{C}^* \left(\sup_{s \in \mathcal{S}_0} |\gamma(s) - \gamma_0(s)| \frac{\log n}{nb_n} \right)^{1/2}
\end{aligned}$$

almost surely when $n^{1-2\epsilon} b_n^2 \to \varrho < \infty$.

Proof of Lemma B.3 We only prove the first results for $T_n(\gamma; s)$ because the proof for $\overline{T}_n(\gamma; s)$ is identical. We define

$$\phi_{3n} = \|\gamma - \gamma_0\|_\infty \frac{\log n}{nb_n},$$

where $\|\gamma-\gamma_0\|_\infty = \sup_{s\in\mathcal{S}_0}|\gamma(s)-\gamma_0(s)|$, which is bounded since $\gamma(s)\in\Gamma$, a compact set, for any s. In addition, when $\|\gamma-\gamma_0\|_\infty=0$, $T_n(\gamma;s)=0$ and hence the result trivially holds. So we suppose $\|\gamma-\gamma_0\|_\infty>0$ without loss of generality. Similar to the proof of Lemma A.3, we let $\tau_n=(n\log n)^{1/(4+2\varphi)}$ with $\varphi>0$ given in Assumption A-(v) and

$$T_n^\tau(\gamma,s)=\frac{1}{nb_n}\sum_{i\in\Lambda_n}\left(c_0^\top x_i\right)^2|\Delta_i(\gamma;s)|\,K_i(s)\,\mathbf{1}_{\tau_n}, \tag{B.47}$$

where $\Delta_i(\gamma;s)=\mathbf{1}_i(\gamma(s))-\mathbf{1}_i(\gamma_0(s))$ and $\mathbf{1}_{\tau_n}=\mathbf{1}[(c_0^\top x_i)^2\le\tau_n]$. The triangular inequality gives that

$$\begin{aligned}\sup_{s\in\mathcal{S}_0}|T_n(\gamma;s)-\mathbb{E}[T_n(\gamma;s)]| &\le \sup_{s\in\mathcal{S}_0}|T_n^\tau(\gamma;s)-T_n(\gamma;s)| \\ &\quad+\sup_{s\in\mathcal{S}_0}|\mathbb{E}[T_n^\tau(\gamma;s)]-\mathbb{E}[T_n(\gamma;s)]| \\ &\quad+\sup_{s\in\mathcal{S}_0}|T_n^\tau(\gamma;s)-\mathbb{E}[T_n^\tau(\gamma;s)]| \\ &\equiv P_{T1}+P_{T2}+P_{T3},\end{aligned}$$

and we bound each of the three terms as follows.

First, we show $P_{T1}=0$ almost surely if n is sufficiently large. By Markov's and Hölder's inequalities,

$$\mathbb{P}\left(\left(c_0^\top x_i\right)^2|\Delta_i(\gamma;s)|>\tau_n\right)\le C\tau_n^{-(4+2\varphi)}\mathbb{E}\left[\left\|x_i^2\right\|^{2(2+\varphi)}\right]\le C'(n\log n)^{-1}$$

for some $C,C'<\infty$ from Assumption A-(v) and the fact that $|\Delta_i(\gamma;s)|\le 1$. Then, as in the proof of Lemma A.3, the Borel-Cantelli lemma implies that $(c_0^\top x_n)^2|\Delta_n(\gamma;s)|\le\tau_n$ almost surely for sufficiently large n. Since $\tau_n\to\infty$, we have $(c_0^\top x_i)^2|\Delta_i(\gamma;s)|\le\tau_n$ almost surely for all $i\in\Lambda_n$ with sufficiently large n. The desired results hence follows.

Second, we show $P_{T2}\le C^*\phi_{3n}^{1/2}$ almost surely for some $C^*<\infty$ if n is sufficiently large. For any $s\in\mathcal{S}_0$,

$$\begin{aligned}&|\mathbb{E}[T_n^\tau(\gamma;s)]-\mathbb{E}[T_n(\gamma;s)]| \\ \le\ & b_n^{-1}\mathbb{E}\left[\left|\left(c_0^\top x_i\right)^2\mathbf{1}\left[\min\{\gamma_0(s),\gamma(s)\}<q_i\le\max\{\gamma_0(s),\gamma(s)\}\right]K_i(s)\left(1-\mathbf{1}_{\tau_n}\right)\right|\right] \\ \le\ & \int\int_{\min\{\gamma_0(s),\gamma(s)\}}^{\max\{\gamma_0(s),\gamma(s)\}}\mathbb{E}\left[\left(c_0^\top x_i\right)^2\left(1-\mathbf{1}_{\tau_n}\right)|q,s+b_nt\right]f(q,s+b_nt)K(t)dqdt \\ \le\ & \tau_n^{-(3+2\varphi)}\int\int_{\min\{\gamma_0(s),\gamma(s)\}}^{\max\{\gamma_0(s),\gamma(s)\}}\mathbb{E}\left[\left(c_0^\top x_i\right)^{2(4+2\varphi)}|q,s+b_nt\right]f(q,s+b_nt)K(t)dqdt \\ \le\ & C\tau_n^{-(3+2\varphi)}\|\gamma-\gamma_0\|_\infty\end{aligned}$$

for some $C<\infty$, where $\mathbb{E}[(c_0^\top x_i)^{2(4+2\varphi)}|q,s]f(q,s)$ is uniformly bounded over (q,s) by Assumptions A-(v) and (vii); and we use the inequality

$$\int_{|a|>\tau_n}af_A(a)\,da\le\tau_n^{-(3+2\varphi)}\int_{|a|>\tau_n}|a|^{4+2\varphi}f_A(a)\,da\le\tau_n^{-(3+2\varphi)}\mathbb{E}\left[A^{4+2\varphi}\right]$$

for a generic random variable A. Hence, the desired result follows since

$$\frac{\tau_n^{-(3+2\varphi)} \|\gamma-\gamma_0\|_\infty}{\phi_{3n}^{1/2}} = \|\gamma-\gamma_0\|_\infty^{1/2} \frac{b_n^{1/2}}{n^{\frac{3+2\varphi}{4+2\varphi}-\frac{1}{2}} (\log n)^{\frac{3+2\varphi}{4+2\varphi}+\frac{1}{2}}} = o(1),$$

where $\|\gamma-\gamma_0\|_\infty$ is bounded and $(3+2\varphi)/(4+2\varphi)-(1/2)>0$.

Finally, we show $P_{T3} \leq C^* \phi_{3n}^{1/2}$ almost surely for some $C^* < \infty$ if n is sufficiently large, which follows similarly as the proof of Lemma A.4. To this end, we partition the compact $\mathcal{S}_0$ into m_n-number of intervals $\mathcal{I}_k = [s_k, s_{k+1})$ for $k = 1, \ldots, m_n$. We choose the integer $m_n > n$ such that $m_n = O(\tau_n n^{(1+\varphi)/(4+2\varphi)}/(b_n \phi_{3n}^{1/2}))$ and $|s_{k+1}-s_k| \leq C/m_n$ for all k and for some $C<\infty$. Note that $m_n/n = C' n^{(1-2\epsilon)/4} / \left[(n^{1-2\epsilon} b_n^2)^{1/4} (\log n)^{1/2-1/(4+2\varphi)}\right] > 1$ for sufficiently large n and C' provided $n^{1-2\epsilon} b_n^2 \to \varrho < \infty$. In addition, since we let $\gamma(\cdot)$ be a cadlag and piecewise constant function with at most n discontinuity points, which is less than m_n, Theorem 28.2 in Davidson (1994) entails that we can choose these finite partitions such that

$$\sup_{s \in \mathcal{I}_k} |\gamma(s) - \gamma(s_k)| = 0 \tag{B.48}$$

for each k. Then we have

$$\begin{aligned}
\sup_{s \in \mathcal{S}_0} \left| T_n^\tau(\gamma; s) - \mathbb{E}\left[T_n^\tau(\gamma; s)\right]\right| &\leq \max_{1 \leq k \leq m_n} \sup_{s \in \mathcal{I}_k} \left|T_n^\tau(\gamma; s) - T_n^\tau(\gamma; s_k)\right| \\
&\quad + \max_{1 \leq k \leq m_n} \sup_{s \in \mathcal{I}_k} \left|\mathbb{E}\left[T_n^\tau(\gamma; s)\right] - \mathbb{E}\left[T_n^\tau(\gamma; s_k)\right]\right| \\
&\quad + \max_{1 \leq k \leq m_n} \left|T_n^\tau(\gamma; s_k) - \mathbb{E}\left[T_n^\tau(\gamma; s_k)\right]\right| \\
&\equiv \Psi_{T1} + \Psi_{T2} + \Psi_{T3}.
\end{aligned}$$

Below we show Ψ_{T1}, Ψ_{T2}, and Ψ_{T3} are all $O_{a.s.}(\phi_{3n}^{1/2})$.

Part 1: Ψ_{T1} and Ψ_{T2} are both $o_{a.s.}(\phi_{3n}^{1/2})$. Similarly as Ψ_{M1} term in Lemma A.3, we first decompose $|T_n^\tau(\gamma; s) - T_n^\tau(\gamma; s_k)| \leq T_{1n}^\tau(\gamma; s, s_k) + T_{2n}^\tau(\gamma; s, s_k)$, where

$$\begin{aligned}
T_{1n}^\tau(\gamma; s, s_k) &= \frac{1}{nb_n} \sum_{i \in \Lambda_n} \left(c_0^\top x_i\right)^2 \left|\Delta_i(\gamma; s) - \Delta_i(\gamma; s_k)\right| K_i(s_k) \mathbf{1}_{\tau_n}, \\
T_{2n}^\tau(\gamma; s, s_k) &= \frac{1}{nb_n} \sum_{i \in \Lambda_n} \left(c_0^\top x_i\right)^2 \left|\Delta_i(\gamma; s)\right| \left|K_i(s) - K_i(s_k)\right| \mathbf{1}_{\tau_n}.
\end{aligned}$$

Since $K_i(\cdot)$ is bounded from Assumption A-(x) and we only consider $x_i^2 \leq \tau_n$,

$$\begin{aligned}
& T_{1n}^\tau(\gamma; s, s_k) \\
\leq\ & \frac{1}{nb_n} \sum_{i \in \Lambda_n} \left(c_0^\top x_i\right)^2 \mathbf{1}\left[\min\{\gamma_0(s), \gamma_0(s_k)\} < q_i \leq \max\{\gamma_0(s), \gamma_0(s_k)\}\right] K_i(s_k) \mathbf{1}_{\tau_n} \\
& + \frac{1}{nb_n} \sum_{i \in \Lambda_n} \left(c_0^\top x_i\right)^2 \mathbf{1}\left[\min\{\gamma(s), \gamma(s_k)\} < q_i \leq \max\{\gamma(s), \gamma(s_k)\}\right] K_i(s_k) \mathbf{1}_{\tau_n} \\
\leq\ & \left\{ \frac{C_1 \tau_n}{b_n} \mathbb{P}\left(\min\{\gamma_0(s), \gamma_0(s_k)\} < q_i \leq \max\{\gamma_0(s), \gamma_0(s_k)\}\right) \right. \\
& \left. + \frac{C_1 \tau_n}{b_n} \mathbb{P}\left(\min\{\gamma(s), \gamma(s_k)\} < q_i \leq \max\{\gamma(s), \gamma(s_k)\}\right) \right\} (1 + o_{a.s.}(1))
\end{aligned}$$

$$
\begin{aligned}
&\leq C_1' \tau_n b_n^{-1} \sup_{s\in\mathcal{I}_k} |s - s_k| + 0 \\
&\leq C_1' \tau_n b_n^{-1} m_n^{-1} \\
&\leq C_1'' n^{-(1+\varphi)/(4+2\varphi)} \phi_{3n}^{1/2}
\end{aligned}
$$

for $C_1, C_1', C_1'' < \infty$, where the second second equality is by the uniform almost sure law of large numbers for random fields (e.g., Jenish and Prucha (2009), Theorem 2); the third inequality is since $\gamma_0(\cdot)$ is continuously differentiable, q_i is continuous, and (B.48). Hence, $T_{1n}^{\tau}(\gamma; s, s_k) = o_{a.s.}(\phi_{3n}^{1/2})$, which holds uniformly in $s \in \mathcal{I}_k$ and $k \in \{1, \ldots, m_n\}$. Similarly, since $K(\cdot)$ is Lipschitz from Assumption A-(x) and $|\Delta_i(\gamma; s)| \leq 1$,

$$
\begin{aligned}
T_{2n}^{\tau}(\gamma; s, s_k) &\leq C_2 \frac{\tau_n}{nb_n} \sum_{i\in\Lambda_n} |K_i(s) - K_i(s_{k_2})| \\
&= C_2 \tau_n \int \left| K(t) - K\left(t + \frac{s - s_{k2}}{b_n}\right)\right| f(s + tb_n)\, dt \\
&\leq C_2' \frac{\tau_n}{b_n} |s - s_{k_2}| \leq \frac{C_2'' \tau_n}{b_n m_n} = o_{a.s.}\left(\phi_{3n}^{1/2}\right)
\end{aligned}
$$

for some $C_2, C_2', C_2'' < \infty$, uniformly in s and k. Hence, $\Psi_{T1} = o_{a.s.}(\phi_{3n}^{1/2})$ and we can readily verify that $\Psi_{T2} = o_{a.s.}(\phi_{3n}^{1/2})$ similarly.

Part 2: $\Psi_{T3} = O_{a.s.}(\phi_{3n}^{1/2})$. We let

$$
Z_i^{\tau}(s) = (nb_n)^{-1}\left\{(c_0^\top x_i)^2 \Delta_i(\gamma; s) K_i(s) \mathbf{1}_{\tau_n} - \mathbb{E}[(c_0^\top x_i)^2 \Delta_i(\gamma; s) K_i(s) \mathbf{1}_{\tau_n}]\right\}
$$

and apply the similar proof as $\Psi_{\Delta M3}$ in Lemma A.4. In particular, we construct the block $\mathcal{B}^{[1]}(s_k)$ in the same fashion as (B.20). Then, it suffices to show $\max_{1\leq k\leq m_n} \left|\mathcal{B}^{[1]}(s_k)\right| = O_{a.s.}(\phi_{3n}^{1/2})$ as $n \to \infty$. Using the same notations as in Lemma A.4, by the uniform almost sure law of large numbers for random fields, we have that for any $t = 1, \ldots, r$ and $s \in \mathcal{S}_0$,

$$
|U_t(s)| \leq \frac{C_3 w^2 \tau_n}{nb_n}\left(\frac{1}{w^2}\sum_{i_1=2j_1w+1}^{(2j_1+1)w} \sum_{i_2=2j_2w+1}^{(2j_2+1)w} |\Delta_i(\gamma; s)|\right) \leq \frac{C_3 w^2 \tau_n \|\gamma - \gamma_0\|_\infty}{nb_n} \tag{B.49}
$$

almost surely from (B.19), for some $C_3 < \infty$. We also approximate $\{U_t(s)\}_{t=1}^r$ by a version of independent random variables $\{U_t^*(s)\}_{t=1}^r$ that satisfies

$$
\sum_{t=1}^r \mathbb{E}\left[|U_t^*(s) - U_t(s)|\right] \leq rC_3 (nb_n)^{-1} w^2 \tau_n \|\gamma - \gamma_0\|_\infty \alpha_{w^2,w^2}(w).
$$

Then, similar to (B.24), for some positive $C^* < \infty$,

$$
\begin{aligned}
\mathbb{P}\left(\max_{1\leq k\leq m_n}\left|\mathcal{B}^{[1]}(s_k)\right| > C^*\phi_{3n}^{1/2}\right) &\leq m_n \sup_{s\in\mathcal{S}_0} \mathbb{P}\left(\sum_{t=1}^r |U_t^*(s) - U_t(s)| > C^*\phi_{3n}^{1/2}\right) \\
&\quad + m_n \sup_{s\in\mathcal{S}_0} \mathbb{P}\left(\left|\sum_{t=1}^r U_t^*(s)\right| > C^*\phi_{3n}^{1/2}\right) \\
&\equiv \widetilde{P}_{U1} + \widetilde{P}_{U2}.
\end{aligned}
$$

For $\widetilde{P}_{U1}$,

$$
\begin{aligned}
\widetilde{P}_{U1} &\leq m_n \frac{rC_3 (nb_n)^{-1} w^2 \tau_n \|\gamma-\gamma_0\|_\infty \alpha_{w^2,w^2}(w)}{C^* \phi_{3n}^{1/2}} \\
&\leq C_3' \frac{\tau_n^2 n^{(1+\varphi)/(4+2\varphi)} \|\gamma-\gamma_0\|_\infty}{b_n^2 \phi_{3n}} \exp(-C_3'' n^{\kappa_1}) \\
&\leq C_3''' \exp(-C_3'' n^{\kappa_1}) \left(\frac{\log n}{n^{1-2\epsilon} b_n}\right) \frac{n^{\kappa_2}}{(\log n)^{\kappa_3}}
\end{aligned}
$$

for some $\kappa_1, \kappa_2, \kappa_3 > 0$ and $C_3', C_3'', C_3''' < \infty$. Hence $\widetilde{P}_{U1} \to 0$ as $n \to \infty$, since $\log n/(n^{1-2\epsilon} b_n) \to 0$ and the exponential term in the last inequality diminishes faster than the polynomial order.

For $\widetilde{P}_{U2}$, using the same argument as (B.3) in Lemma A.1, we can show that

$$
\mathbb{E}\left[U_t^*(s)^2\right] = \sum_{\substack{1 \leq i_1 \leq w \\ 1 \leq i_2 \leq w}} \mathbb{E}\left[Z_i^\tau(s)^2\right] + \sum_{\substack{i \neq j \\ 1 \leq i_1, i_2 \leq w \\ 1 \leq j_1, j_2 \leq w}} Cov\left[Z_i^\tau(s), Z_j^\tau(s)\right] \leq \frac{C_4 w^2}{n^2 b_n} \|\gamma-\gamma_0\|_\infty
$$

for some $C_4 < \infty$, which does not depend on s given Assumptions A-(v) and (x). We now choose an integer w such that

$$
\begin{aligned}
w &= (nb_n / (C_w \tau_n \lambda_n))^{1/2}, \\
\lambda_n &= (nb_n \log n)^{1/2}
\end{aligned}
$$

for some large positive constant C_w. Note that, substituting λ_n and τ_n into w gives

$$
w = O\left(\left[\frac{n^{\varphi/(2+\varphi)}}{(\log n)^{(4+\varphi)/(2+\varphi)}} \left(\frac{nb_n^2}{\log n}\right)\right]^{1/8}\right),
$$

which diverges as $n \to \infty$ for $\varphi > 0$ and from Assumption A-(ix). From (B.49), we have $|\lambda_n U_t^*(s)/\|\gamma-\gamma_0\|_\infty^{1/2}| < 1/2$ by choosing C_w large enough, and hence

$$
\begin{aligned}
\sup_{s \in \mathcal{S}_0} \mathbb{P}\left(\left|\sum_{t=1}^r U_t^*(s)\right| > C^* \phi_{3n}^{1/2}\right) &= \sup_{s \in \mathcal{S}_0} \mathbb{P}\left(\left|\sum_{t=1}^r \frac{\lambda_n U_t^*(s)}{\|\gamma-\gamma_0\|_\infty^{1/2}}\right| > C^* \left(\frac{\log n}{nb_n}\right)^{1/2}\right) \\
&\leq 2\exp\left(-C^* \lambda_n \left(\frac{\log n}{nb_n}\right)^{1/2} + \frac{C_4 \lambda_n^2 r w^2}{n^2 b_n}\right) \\
&= 2\exp\left(-C^* \lambda_n \left(\frac{\log n}{nb_n}\right)^{1/2} + C_4 \lambda_n^2 (nb_n)^{-1}\right) \\
&= 2\exp\left(-C^* \log n + C_4' \log n\right)
\end{aligned}
$$

for some $C_4, C_4' < \infty$ as in (B.26) and (B.27). It follows that

$$
\widetilde{P}_{U2} = m_n \sup_{s \in \mathcal{S}_0} \mathbb{P}\left(\left|\sum_{t=1}^r U_t^*(s)\right| > C^* \phi_{3n}^{1/2}\right) \leq \frac{2m_n}{n^{C^*-C_4'}} \leq C_5 \left(\frac{\log n}{n^{1-2\epsilon} b_n}\right)^{3/2} \frac{1}{(\log n)^{\kappa_4} n^{\kappa_5}}
$$

for some $C_5 < \infty$, $\kappa_4 = 1 - (1/(4+2\varphi)) > 1$, and $\kappa_5 = (C^* - C_4') - 1 - ((1-2\epsilon)/2) > 1$ by choosing C^* sufficiently large (e.g., $C^* > C_4' + 5/2$). Therefore, $\widetilde{P}_{U2} \leq O(n^{-\kappa_5}) \to 0$ as $n \to \infty$.

Since $\widetilde{P}_{U1}+\widetilde{P}_{U2}=O(n^{-c})$ for some $c>1$, we have $\sum_{n=1}^{\infty}\mathbb{P}(\max_{1\leq k\leq m_n}\left|\mathcal{B}^{[1]}(s_k)\right|>C^*\phi_{3n}^{1/2})<\infty$ and hence we obtain the desired result by the Borel-Cantelli lemma. ■

Lemma B.4 *There exists some constant C_L such that for any $\gamma(\cdot)\in\mathcal{G}_n(\mathcal{S}_0;\Gamma)$ and any $j=1,\ldots,\dim(x)$*

$$\sup_{s\in\mathcal{S}_0}|L_{nj}(\gamma;s)|\leq C_L\left(\sup_{s\in\mathcal{S}_0}|\gamma(s)-\gamma_0(s)|\log n\right)^{1/2}$$

almost surely when $n^{1-2\epsilon}b_n^2\to\varrho<\infty$.

Proof of Lemma B.4 The proof is similar to that in Lemma B.3, and we only highlight the different parts. We assume x_i is a scalar, so as $L_n(\gamma;s)$. As in (B.47), we let

$$L_n^{\tau}(\gamma;s)=\frac{1}{\sqrt{nb_n}}\sum_{i\in\Lambda_n}x_iu_i\Delta_i(\gamma;s)K_i(s)\mathbf{1}_{\tau_n},$$

where $\Delta_i(\gamma;s)=\mathbf{1}_i(\gamma(s))-\mathbf{1}_i(\gamma_0(s))$ and $\mathbf{1}_{\tau_n}=\mathbf{1}[|x_iu_i|\leq\tau_n]$ with $\tau_n=(n\log n)^{1/(4+2\varphi)}$. Since $\mathbb{E}[L_n^{\tau}(\gamma,s)]=0$, we write

$$\begin{aligned}\sup_{s\in\mathcal{S}_0}|L_n(\gamma;s)| &\leq \sup_{s\in\mathcal{S}_0}|L_n^{\tau}(\gamma;s)-L_n(\gamma;s)|+\sup_{s\in\mathcal{S}_0}|L_n^{\tau}(\gamma;s)|\\ &\equiv P_{L1}+P_{L2}.\end{aligned}$$

Using the same argument as P_{T1} in the proof of Lemma B.3, we have

$$\mathbb{P}\left(|x_iu_i|\,|\Delta_i(\gamma;s)|>\tau_n\right)\leq C\tau_n^{-(4+2\varphi)}\mathbb{E}\left[\|x_iu_i\|^{2(2+\varphi)}\right]\leq C'(n\log n)^{-1}$$

for some $C,C'<\infty$. Then the Borel-Cantelli lemma implies that $|x_iu_i|\,|\Delta_n(\gamma;s)|\leq\tau_n$ almost surely for sufficiently large n. Since $\tau_n\to\infty$, we have $|x_iu_i|\,|\Delta_i(\gamma;s)|\leq\tau_n$ almost surely for all $i\in\Lambda_n$ with sufficiently large n, which yields $P_{L1}=0$ almost surely for a sufficiently large n.

For P_{L2}, we let $\widetilde{\phi}_{3n}=\|\gamma-\gamma_0\|_{\infty}\log n$ and write

$$\begin{aligned}\sup_{s\in\mathcal{S}_0}|L_n^{\tau}(\gamma;s)| &\leq \max_{1\leq k\leq m_n}\sup_{s\in\mathcal{I}_k}|L_n^{\tau}(\gamma;s)-L_n^{\tau}(\gamma;s_k)|+\max_{1\leq k\leq m_n}|L_n^{\tau}(\gamma;s_k)|\\ &\equiv \Psi_{L1}+\Psi_{L2},\end{aligned}$$

for some integer $m_n=O(\tau_nn^{(3+2\varphi)/(4+2\varphi)}/(b_n\widetilde{\phi}_{3n}^{1/2}))$, where $m_n/n>1$ for sufficiently large n. We let $Z_i^{\tau}(s)=(nb_n)^{-1/2}x_iu_i\Delta_i(\gamma;s)K_i(s)\mathbf{1}_{\tau_n}$, and we choose $w=((nb_n)/(C_w\tau_n\lambda_n))^{1/2}$ for some large positive constant C_w and $\lambda_n=(\log n)^{1/2}$. Then, the rest of the proof follows similarly as bounding P_{T3} in the proof of Lemma B.3. ■

Proof of Lemma A.13 We first show (A.23). We consider the case with $\sup_{s\in\mathcal{S}_0}(\gamma(s)-\gamma_0(s))>0$, and the other direction can be shown symmetrically. We suppose n is large enough so that $\overline{r}\phi_{2n}\leq\overline{C}$ for some $\overline{r},\overline{C}\in(0,\infty)$ and $\sup_{s\in\mathcal{S}_0}(\gamma(s)-\gamma_0(s))\in\left[\overline{r}\phi_{2n},\overline{C}\right]$. We also let

$$\underline{\ell}=\inf_{s\in\mathcal{S}_0}\underline{\ell}_D(s)>0$$

where $\underline{\ell}_D(s)$ is defined in (B.28). Then, from (B.29), we have

$$\sup_{s\in\mathcal{S}_0}\mathbb{E}\left[T_n(\gamma;s)\right]\geq\underline{\ell}\sup_{s\in\mathcal{S}_0}(\gamma(s)-\gamma_0(s)).\tag{B.50}$$

For any $\varepsilon > 0$ and for any $\gamma(\cdot)$ such that $\sup_{s\in\mathcal{S}_0}(\gamma(s) - \gamma_0(s)) \in [\overline{r}\phi_{2n}, \overline{C}]$, Lemma B.3 and (B.50) imply that when n is sufficiently large,

$$
\begin{aligned}
& \frac{\sup_{s\in\mathcal{S}_0} T_n(\gamma; s)}{\sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)|} \\
\geq\ & \frac{\sup_{s\in\mathcal{S}_0} \mathbb{E}[T_n(\gamma; s)] - \sup_{s\in\mathcal{S}_0} |T_n(\gamma; s) - \mathbb{E}[T_n(\gamma; s)]|}{\sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)|} \\
\geq\ & \frac{\sup_{s\in\mathcal{S}_0} \mathbb{E}[T_n(\gamma; s)]}{\sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)|} - \frac{\left(\sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)| (\log n/n)\right)^{1/2}}{\sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)|} \\
\geq\ & \underline{\ell} - \frac{(\log n/n)^{1/2}}{\overline{r}\phi_{2n}^{1/2}} \geq \underline{\ell} - \overline{r}^{-1} n^{-\epsilon}.
\end{aligned}
$$

Since $\underline{\ell} > 0$ does not depend on $\gamma(\cdot)$ and $\overline{r}^{-1}n^{-\epsilon} \to 0$ as $n \to \infty$, we thus can find $C_T < \infty$ such that

$$
\mathbb{P}\left(\inf_{\substack{\{\gamma(\cdot)\in\mathcal{G}_n(\mathcal{S}_0;\Gamma): \\ \overline{r}\phi_{2n} < \sup_{s\in\mathcal{S}_0}|\gamma(s)-\gamma_0(s)| < \overline{C}\}}} \frac{\sup_{s\in\mathcal{S}_0} T_n(\gamma; s)}{\sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)|} < C_T(1-\eta) \right) \leq \varepsilon.
$$

for any $\varepsilon, \eta > 0$. The proof for (A.24) is similar and hence omitted.

For (A.25), we present the case of scalar x_i and so is $L_n(\gamma; s)$, for expositional simplicity. We set γ_g for $g = 1, 2, ..., \overline{g} + 1$ such that, for any $s \in \mathcal{S}_0$, $\gamma_g(s) = \gamma_0(s) + 2^{g-1}\overline{r}\phi_{2n}$, where $\overline{g}$ is an integer satisfying $\sup_{s\in\mathcal{S}_0}(\gamma_{\overline{g}}(s) - \gamma_0(s)) = 2^{\overline{g}-1}\overline{r}\phi_{2n} \leq \overline{C}$ and $\sup_{s\in\mathcal{S}_0}(\gamma_{\overline{g}+1}(s) - \gamma_0(s)) > \overline{C}$. Then Lemma B.4 yields that for any $\eta > 0$,

$$
\begin{aligned}
& \mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \frac{\sup_{s\in\mathcal{S}_0} |L_n(\gamma_g; s)|}{\sqrt{a_n} \sup_{s\in\mathcal{S}_0} (\gamma_g(s) - \gamma_0(s))} > \frac{\eta}{4} \right) \qquad \text{(B.51)} \\
\leq\ & \sum_{g=1}^{\overline{g}} \mathbb{P}\left(\frac{\sup_{s\in\mathcal{S}_0} |L_n(\gamma_g; s)|}{\sqrt{a_n} \sup_{s\in\mathcal{S}_0} (\gamma_g(s) - \gamma_0(s))} > \frac{\eta}{4} \right) \\
\leq\ & \frac{4}{\eta} \sum_{g=1}^{\overline{g}} \frac{C_L (\phi_{2n} \log n)^{1/2}}{\sqrt{a_n} 2^{g-1} \overline{r}\phi_{2n}} \\
\leq\ & \frac{C_L'}{\eta\overline{r}} \sum_{g=1}^{\infty} \frac{1}{2^{(g-1)}}
\end{aligned}
$$

for some $C_L, C_L' < \infty$. This probability is arbitrarily close to zero if $\overline{r}$ is chosen large enough. Following a similar discussion after (B.34), this result also provides the maximal (or sharp) rate of ϕ_{2n} as $\log n/a_n$ because we need $(\log n/a_n)/\phi_{2n} = O(1)$ but $\phi_{2n} \to 0$ as $\log n/a_n \to 0$ with $n \to \infty$. For a given g, we define Γ_g as the collection of $\gamma(\cdot) \in \mathcal{G}_n(\mathcal{S}_0; \Gamma)$ satisfying $\overline{r}2^{g-1}\phi_{2n} < \sup_{s\in\mathcal{S}_0} |\gamma(s) - \gamma_0(s)| < \overline{r}2^g\phi_{2n}$. By a similar argument as (B.51) and Lemma B.4, we have

$$
\mathbb{P}\left(\max_{1\leq g\leq \overline{g}} \sup_{\gamma\in\Gamma_g} \frac{\sup_{s\in\mathcal{S}_0} |L_n(\gamma; s) - L_n(\gamma_g; s)|}{\sqrt{a_n} \sup_{s\in\mathcal{S}_0} (\gamma_g(s) - \gamma_0(s))} > \frac{\eta}{4} \right) \leq \frac{C_L''}{\eta\overline{r}} \qquad \text{(B.52)}
$$

for some $C_L'' < \infty$, which is arbitrarily close to zero if $\overline{r}$ is chosen large enough. From (B.36), and by combining (B.51) and (B.52), we thus have

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\substack{\{\gamma(\cdot)\in\mathcal{G}_n(\mathcal{S}_0;\Gamma):\\ \overline{r}\phi_{2n}<\sup_{s\in\mathcal{S}_0}|\gamma(s)-\gamma_0(s)|<\overline{C}\}}} \frac{\sup_{s\in\mathcal{S}_0}|L_n(\gamma;s)|}{\sqrt{a_n}\sup_{s\in\mathcal{S}_0}(\gamma(s)-\gamma_0(s))}>\eta\right)\\
\leq\ & \mathbb{P}\left(2\max_{1\leq g\leq\overline{g}}\frac{\sup_{s\in\mathcal{S}_0}\left|L_n\left(\gamma_g;s\right)\right|}{\sqrt{a_n}\sup_{s\in\mathcal{S}_0}\left(\gamma_g(s)-\gamma_0(s)\right)}>\frac{\eta}{2}\right)\\
&+\mathbb{P}\left(2\max_{1\leq g\leq\overline{g}}\sup_{\gamma\in\Gamma_g}\frac{\sup_{s\in\mathcal{S}_0}\left|L_n(\gamma;s)-L_n\left(\gamma_g;s\right)\right|}{\sqrt{a_n}\sup_{s\in\mathcal{S}_0}\left(\gamma_g(s)-\gamma_0(s)\right)}>\frac{\eta}{2}\right)\\
\leq\ & \varepsilon
\end{aligned}$$

for any $\varepsilon,\eta>0$ if $\overline{r}$ is chosen sufficiently large. ■

Proof of Lemma A.14 We prove $\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\gamma_0(s)|=o_p(1)$. From (B.37), we have

$$n^{\epsilon}\sup_{s\in\mathcal{S}_0}||\widehat{\theta}(\widehat{\gamma}(s))-\theta_0||\leq\left(\inf_{s\in\mathcal{S}_0}|\Theta_{A1}(s)|\right)^{-1}\left\{\sup_{s\in\mathcal{S}_0}|\Theta_{A2}(s)|+\sup_{s\in\mathcal{S}_0}|\Theta_{A3}(s)|\right\}.$$

Hence, given Lemma A.3 and the standard uniform convergence result of the kernel estimators, $n^{\epsilon}\sup_{s\in\mathcal{S}_0}||\widehat{\theta}(\widehat{\gamma}(s))-\theta_0||=o_p(1)$ can be obtained similarly as the proof of Lemma A.7, provided that we have $\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\gamma_0(s)|\to_p 0$ as $n\to\infty$. Recall that $\widehat{\gamma}(s)$ is the minimizer of $\Upsilon_n(\gamma;s)$ in (A.5) and $\gamma_0(s)$ is the minimizer of $\Upsilon_0(\gamma;s)$ in (A.7) for any given $s\in\mathcal{S}_0$. See Lemma A.5 for the definitions of $\Upsilon_n(\gamma;s)$ and $\Upsilon_0(\gamma;s)$.

Suppose $\widehat{\gamma}(s)$ is not uniformly consistent, implying that there exist $\eta>0$ and $\varepsilon>0$ such that for any $N\in\mathbb{N}$, there exists $n>N$ satisfying

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\gamma_0(s)|>\eta\right)\\
=\ &\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}(\widehat{\gamma}(s)-\gamma_0(s))>\eta\right)+\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}(\widehat{\gamma}(s)-\gamma_0(s))<-\eta\right)>\varepsilon
\end{aligned}$$

or simply

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}(\widehat{\gamma}(s)-\gamma_0(s))>\eta\right)>\varepsilon \tag{B.53}$$

without loss of generality. From (A.8), we can define $C\in(0,\infty)$ such that

$$\inf_{s\in\mathcal{S}_0}\frac{\partial\Upsilon_0(\gamma_0(s);s)}{\partial\gamma}>C>0,$$

and hence the mean value theorem yields

$$\begin{aligned}
\Upsilon_0(\widehat{\gamma}(s),s)-\Upsilon_0(\gamma_0(s),s) &= \frac{\partial\Upsilon_0(\widetilde{\gamma}(s),s)}{\partial\gamma}(\widehat{\gamma}(s)-\gamma_0(s))\\
&> C(\widehat{\gamma}(s)-\gamma_0(s))
\end{aligned}$$

for sufficiently large n, where $\widetilde{\gamma}(s)$ is between $\widehat{\gamma}(s)$ and $\gamma_0(s)$. Therefore,

$$\begin{aligned}
& \mathbb{P}\left(\sup_{s\in\mathcal{S}_0}\{\Upsilon_0(\widehat{\gamma}(s),s)-\Upsilon_0(\gamma_0(s),s)\}>C\eta\right) \\
> \ & \mathbb{P}\left(\inf_{s\in\mathcal{S}_0}\frac{\partial\Upsilon_0(\gamma_0(s);s)}{\partial\gamma}\sup_{s\in\mathcal{S}_0}(\widehat{\gamma}(s)-\gamma_0(s))>C\eta\right) \\
= \ & \mathbb{P}\left(\sup_{s\in\mathcal{S}_0}(\widehat{\gamma}(s)-\gamma_0(s))>\eta\right)>\varepsilon
\end{aligned} \tag{B.54}$$

from (B.53).

However, by construction, $\Upsilon_n(\widehat{\gamma}(s),s)-\Upsilon_n(\gamma_0(s),s)\leq 0$ for every $s\in\mathcal{S}_0$, which implies

$$\sup_{s\in\mathcal{S}_0}\{\Upsilon_n(\widehat{\gamma}(s),s)-\Upsilon_n(\gamma_0(s),s)\}\leq 0 \ \text{ almost surely.} \tag{B.55}$$

Furthermore, using the triangular inequality and the uniform convergence result in Lemma A.3, we can verify that

$$\sup_{(r,s)\in\Gamma\times\mathcal{S}_0}|\Upsilon_n(r,s)-\Upsilon_0(r,s)|\to_p 0 \tag{B.56}$$

as $n\to\infty$ from the proof of Lemma A.5. From (B.55) and (B.56), we thus have

$$\begin{aligned}
& \mathbb{P}\left(\sup_{s\in\mathcal{S}_0}\{\Upsilon_0(\widehat{\gamma}(s),s)-\Upsilon_0(\gamma_0(s),s)\}>C\eta\right) \\
\leq \ & \mathbb{P}\left(\sup_{s\in\mathcal{S}_0}\{\Upsilon_0(\widehat{\gamma}(s),s)-\Upsilon_n(\widehat{\gamma}(s),s)\}>C\eta/3\right) \\
& +\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}\{\Upsilon_n(\widehat{\gamma}(s),s)-\Upsilon_n(\gamma_0(s),s)\}>C\eta/3\right) \\
& +\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}\{\Upsilon_n(\gamma_0(s),s)-\Upsilon_0(\gamma_0(s),s)\}>C\eta/3\right) \\
\leq \ & (\varepsilon^*/3)+(\varepsilon^*/3)+(\varepsilon^*/3)=\varepsilon^*
\end{aligned}$$

for any $\varepsilon^*>0$ if n is sufficiently large. It contradicts to (B.54) by choosing $\varepsilon^*\leq\varepsilon$, hence the uniform consistency should hold. ■

Proof of Lemma A.15 We prove $\Xi_{\beta 2}=o_p(1)$ and $\Xi_{\beta 3}=o_p(1)$. The results for $\Xi_{\delta 2}$ and $\Xi_{\delta 3}$ can be shown symmetrically. For expositional simplicity, we present the case of scalar x_i.

For $\Xi_{\beta 2}$: Note that $\widehat{\gamma}(\cdot)$ belongs to $\mathcal{G}_n(\mathcal{S}_0;\Gamma)$. We define intervals $\mathcal{I}_k$ for $k=1,\ldots,n$, which are centered at the discontinuity points of $\widehat{\gamma}(s)$ with length ℓ_n such that $\ell_n\to 0$ as $n\to\infty$. Without loss of generality, we choose $\ell_n=O(n^{-3})$. Then, we can interpolate on each $\mathcal{I}_k$ and define $\widetilde{\gamma}(s)$ as a smooth version of $\widehat{\gamma}(s)$, which satisfies

$$\mathbb{P}\left(\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\widetilde{\gamma}(s)|>\varepsilon\right)\leq\mathbb{P}\left(\max_{1\leq k\leq n}\sup_{s\in\mathcal{I}_k}|\widehat{\gamma}(s)-\widetilde{\gamma}(s)|>\varepsilon\right)\leq\varepsilon \tag{B.57}$$

for any $\varepsilon>0$, if n is sufficiently large. Since $\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\gamma_0(s)|=o_p(1)$ from Lemma A.14, we have

$$\sup_{s\in\mathcal{S}_0}|\widetilde{\gamma}(s)-\gamma_0(s)|\leq\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\widetilde{\gamma}(s)|+\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s)-\gamma_0(s)|=o_p(1) \tag{B.58}$$

from (B.57).

Now we define

$$G_n(\gamma) = \frac{1}{\sqrt{n}} \sum_{i \in \Lambda_n} x_i u_i \mathbf{1}\left[q_i > \gamma(s_i) + \pi_n\right] \mathbf{1}_{\mathcal{S}_0},$$

and then

$$\begin{aligned}
\Xi_{\beta 2} &= G_n(\widehat{\gamma}) - G_n(\gamma_0) \\
&= \{G_n(\widehat{\gamma}) - G_n(\widetilde{\gamma})\} + \{G_n(\widetilde{\gamma}) - G_n(\gamma_0)\} \\
&\equiv \Psi_{G1} + \Psi_{G2}.
\end{aligned}$$

First, for Ψ_{G1}, let $\Delta_i^{\pi}(\widehat{\gamma}, \widetilde{\gamma}) = \mathbf{1}\left[q_i > \widehat{\gamma}(s_i) + \pi_n\right] - \mathbf{1}\left[q_i > \widetilde{\gamma}(s_i) + \pi_n\right]$. By construction, $|\Delta_i^{\pi}(\widehat{\gamma}, \widetilde{\gamma})| \leq \mathbf{1}[s_i \in \mathcal{I}_k \text{ for some } k]$. Therefore, by the Cauchy-Schwarz inequality and Assumptions A-(v) and A-(viii),

$$\begin{aligned}
\mathbb{E}\left[|\Psi_{G1}|\right] &\leq n^{1/2} \mathbb{E}\left[|x_i u_i| \left|\Delta_i^{\pi}(\widehat{\gamma}, \widetilde{\gamma})\right| \mathbf{1}_{\mathcal{S}_0}\right] \\
&\leq n^{1/2} \mathbb{E}\left[(x_i u_i)^2\right]^{1/2} \mathbb{E}\left[\left(\mathbf{1}[s_i \in \mathcal{I}_k \text{ for some } k] \mathbf{1}_{\mathcal{S}_0}\right)^2\right]^{1/2} \\
&\leq C_1 n^{1/2} \left(\mathbb{P}[s_i \in \mathcal{I}_k \cap \mathcal{S}_0 \text{ for some } k]\right)^{1/2} \\
&\leq C_1' n^{1/2} n^{-3/2} = o(1)
\end{aligned}$$

for some $C_1, C_1' < \infty$. Hence, $\Psi_{G1} = o_p(1)$.

Second, for Ψ_{G2}, we let $\mathbf{1}_\tau = \mathbf{1}[|x_i u_i| \leq \tau]$ for some $\tau < \infty$. Then, for any $\varepsilon_1 > 0$ and $\gamma : \mathcal{S}_0 \mapsto \Gamma$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i \in \Lambda_n} x_i u_i \mathbf{1}\left[q_i > \gamma(s_i) + \pi_n\right] (1 - \mathbf{1}_\tau) \mathbf{1}_{\mathcal{S}_0} > \varepsilon_1\right) \\
\leq\ & \varepsilon_1^{-2} \frac{1}{n} \mathbb{E}\left[\left(\sum_{i \in \Lambda_n} x_i u_i \mathbf{1}\left[q_i > \gamma(s_i) + \pi_n\right] (1 - \mathbf{1}_\tau) \mathbf{1}_{\mathcal{S}_0}\right)^2\right] \\
\leq\ & C \varepsilon_1^{-2} \mathbb{E}\left[(x_i u_i)^2 \mathbf{1}\left[|x_i u_i| > \tau\right]\right] \\
\leq\ & C \varepsilon_1^{-2} \mathbb{E}\left[(x_i u_i)^4\right]^{1/2} \left(\mathbb{P}\left[|x_i u_i| > \tau\right]\right)^{1/2} \\
\leq\ & C \varepsilon_1^{-2} \tau^{-2} \mathbb{E}\left[(x_i u_i)^4\right]
\end{aligned}$$

for some $C < \infty$, where we apply the Markov's and the Cauchy-Schwarz inequalities. From Assumption A-(v), by choosing τ sufficiently large, this probability can be arbitrarily small. Hence,

$$G_n(\gamma) = \frac{1}{\sqrt{n}} \sum_{i \in \Lambda_n} x_i u_i \mathbf{1}\left[q_i > \gamma(s_i) + \pi_n\right] \mathbf{1}_\tau \mathbf{1}_{\mathcal{S}_0} + o_p(1)$$

for sufficiently large n and we simply consider $|x_i u_i| \leq \tau$ almost surely in what follows.

We let $\mathcal{F}^*$ be the class of functions $\{xu\mathbf{1}\left[q > \gamma(s) + \pi_n\right] \text{ for } \gamma \in \mathcal{C}^2[\mathcal{S}_0]\}$, where $\mathcal{C}^2[\mathcal{S}_0]$ denotes the family of twice-continuously differentiable functions defined on $\mathcal{S}_0$. Using Theorem 2.5.6 in der Vaart and Wellner (1996), we establish that $\mathcal{F}^*$ is P-Donsker, which requires three elements: an entropy bound, a maximal inequality, and the chaining argument. For the entropy bound, by Corollaries 2.7.2 and 2.7.3 in der Vaart and Wellner (1996) (with their $r = d = 1$ and $\alpha = 2$),

$\mathcal{F}^*$ has the same bracketing number (up to a constant) as that for the collection of subgraphs of $\mathcal{C}^2[\mathcal{S}_0]$, so that $\log N_{[]}(\varepsilon, \mathcal{F}^*, ||\cdot||_\infty) \leq C\varepsilon^{-1/2}$, where $||\cdot||_\infty$ denotes the uniform norm. For the maximal inequality, since we consider $|xu| \leq \tau$, Corollary 3.3 in Valenzuela-Domíguez, Krebs, and Franke (2017) gives the Bernstein inequality for spatial lattice processes with exponentially decaying α-mixing coefficients. This satisfies the conditions in Lemma 2.2.10 in der Vaart and Wellner (1996), which implies that for any finite collection of functions $\gamma_1, \ldots, \gamma_m \in \mathcal{C}^2[\mathcal{S}_0]$,

$$\mathbb{E}\left[\max_{1\leq k\leq m} G_n(\gamma_k)\right] \leq C'\left(\log(1+m) + \sqrt{\log(1+m)}\right) \tag{B.59}$$

for $C' < \infty$. For the chaining argument, the same analysis in der Vaart and Wellner (1996), pp.131-132 applies with the following two changes: their envelope function F is $|xu|$, which satisfies $\mathbb{E}\left[F^2\right] < \infty$; and their inequality (2.5.5) is implied by (B.59) with $m = \log N_{[]}(\varepsilon, \mathcal{F}^*, ||\cdot||_\infty)$. Note that the spatial dependence only shows up in deriving the maximal inequality but not the entropy or the chaining argument.

Since Donsker implies stochastic equicontinuity, it follows that $G_n(\cdot)$ satisfies, for every positive $\eta_n \to 0$,

$$\sup_{\sup_{s\in\mathcal{S}_0}|\gamma(s)-\gamma'(s)|\leq\eta_n} \left|G_n(\gamma) - G_n(\gamma')\right| \to_p 0$$

as $n \to \infty$. Therefore, $\Psi_{G2} = o_p(1)$ since $\sup_{s\in\mathcal{S}_0}|\widetilde{\gamma}(s) - \gamma_0(s)| = o_p(1)$ from (B.58).

For $\Xi_{\beta3}$: On the event E_n^* that $\sup_{s\in\mathcal{S}_0}|\widehat{\gamma}(s) - \gamma_0(s)| \leq \phi_{2n}$, we have

$$\begin{aligned}
\mathbb{E}\left[|\Xi_{\beta3}|\right] &= \frac{1}{\sqrt{n}}\sum_{i\in\Lambda_n}\mathbb{E}\left[\left|x_i^2\delta_0\right|\mathbf{1}\left[q_i \leq \gamma_0(s_i)\right]\mathbf{1}\left[q_i > \widehat{\gamma}(s_i) + \pi_n\right]\mathbf{1}_{\mathcal{S}_0}\right] \\
&\leq n^{1/2-\epsilon}C\mathbb{E}\left[\mathbf{1}\left[q_i \leq \gamma_0(s_i)\right]\mathbf{1}\left[q_i > \widehat{\gamma}(s_i) + \pi_n\right]\mathbf{1}_{\mathcal{S}_0}\right] \\
&\leq n^{1/2-\epsilon}C\mathbb{E}\left[\mathbf{1}\left[q_i \leq \gamma_0(s_i)\right]\mathbf{1}\left[q_i > \gamma_0(s_i) - \phi_{2n} + \pi_n\right]\mathbf{1}_{\mathcal{S}_0}\right] \\
&= n^{1/2-\epsilon}C\int_{\mathcal{S}_0}\int_{\mathcal{I}(q;s)} f(q,s)dqds
\end{aligned}$$

for some $0 < C < \infty$, where $\mathcal{I}(q;s) = \{q : q \leq \gamma_0(s) \text{ and } q > \gamma_0(s) - \phi_{2n} + \pi_n\}$. Since we define $\pi_n > 0$ such that $\phi_{2n}/\pi_n \to 0$, it holds that $\pi_n - \phi_{2n} > 0$ for sufficiently large n. Therefore, $\mathcal{I}(q;s)$ becomes empty for all s when n is sufficiently large. The desired result follows from Markov's inequality and the fact that $\mathbb{P}(E_n^*) > 1 - \varepsilon$ for any $\varepsilon > 0$. ■

S.2 Additional Simulation Results

This section provides additional simulation results. The data generating process is the same as that in Section 5 in the main text except that $\gamma_0(s) = \sin(s)/2$. Tables S.1 to S.4 below present the analogous results to those in Tables 2 to 5 in the main text. We also plot the averaged $\widehat{\gamma}(s)$ across simulations and the density estimator of $\widehat{\delta}_2 - \delta_{20}$ in Figure S.1. The findings are similar to Figure 3 in the main text.

Table S.1: Bias, RMSE, and Rej. Prob. of the LR Test with i.i.d. Data

	$s = 0.0$				$s = 0.5$				$s = 1.0$			
$n\backslash\delta$	1	2	3	4	1	2	3	4	1	2	3	4
	Bias											
100	-0.07	-0.05	-0.04	-0.04	-0.25	-0.19	-0.14	-0.12	-0.44	-0.33	-0.30	-0.27
200	-0.05	-0.02	-0.04	-0.03	-0.21	-0.14	-0.09	-0.06	-0.36	-0.27	-0.22	-0.17
500	-0.03	-0.03	-0.02	-0.02	-0.14	-0.06	-0.04	-0.03	-0.28	-0.13	-0.11	-0.07
	RMSE											
100	0.27	0.14	0.08	0.06	0.35	0.21	0.12	0.09	0.51	0.37	0.28	0.21
200	0.25	0.08	0.05	0.03	0.30	0.15	0.08	0.05	0.45	0.29	0.20	0.15
500	0.19	0.05	0.02	0.01	0.22	0.08	0.03	0.02	0.37	0.14	0.08	0.05
	Rej. Prob. of the LR test											
100	0.14	0.09	0.07	0.08	0.16	0.09	0.09	0.07	0.27	0.17	0.14	0.13
200	0.10	0.06	0.06	0.07	0.11	0.07	0.06	0.05	0.19	0.10	0.07	0.07
500	0.08	0.04	0.05	0.07	0.07	0.05	0.04	0.05	0.11	0.06	0.03	0.03

Note: Entries are bias and root mean squared error (RMSE) of the estimator $\hat{\gamma}(s)$ and rejection probabilities of the LR test (13) when data are generated from (18) with $\gamma_0(s) = \sin(s)/2$. The dependence structure is given in (19) with $\rho = 0$. The significance level is 5% and the results are based on 1000 simulations.

Table S.2: Bias, RMSE, and Rej. Prob. of the LR Test with Cross-sectionally Correlated Data

	$s=0.0$				$s=0.5$				$s=1.0$			
$n\backslash\delta$	1	2	3	4	1	2	3	4	1	2	3	4
	Bias											
100	-0.04	-0.05	-0.05	-0.07	-0.24	-0.19	-0.15	-0.14	-0.43	-0.37	-0.30	-0.30
200	-0.04	-0.06	-0.05	-0.03	-0.21	-0.12	-0.08	-0.07	-0.40	-0.30	-0.23	-0.19
500	-0.03	-0.02	-0.02	-0.02	-0.16	-0.07	-0.04	-0.03	-0.33	-0.18	-0.11	-0.09
	RMSE											
100	0.29	0.17	0.11	0.09	0.34	0.23	0.16	0.11	0.48	0.38	0.29	0.26
200	0.28	0.13	0.07	0.04	0.34	0.15	0.09	0.06	0.50	0.33	0.20	0.15
500	0.21	0.06	0.03	0.01	0.28	0.10	0.04	0.02	0.42	0.20	0.10	0.06
	Rej. Prob. of the LR test											
100	0.18	0.13	0.09	0.08	0.19	0.12	0.10	0.07	0.33	0.21	0.17	0.13
200	0.14	0.06	0.07	0.06	0.13	0.08	0.05	0.06	0.19	0.12	0.09	0.07
500	0.09	0.06	0.06	0.07	0.10	0.05	0.04	0.05	0.11	0.06	0.04	0.04

Note: Entries are bias and root mean squared error (RMSE) of the estimator $\hat{\gamma}(s)$ and rejection probabilities of the LR test (13) when data are generated from (18) with $\gamma_0(s) = \sin(s)/2$. The dependence structure is given in (19) with $\rho = 1$ and $m = 10$. The significance level is 5% and the results are based on 1000 simulations.

Table S.3: Bias and RMSE of the Coefficient Estimates

	β_{20}				$\beta_{20}+\delta_{20}$				δ_{20}			
$n\backslash\delta$	1	2	3	4	1	2	3	4	1	2	3	4
	Bias											
100	0.07	0.10	0.07	0.05	-0.07	-0.08	-0.06	-0.03	-0.14	-0.17	-0.14	-0.09
200	0.07	0.06	0.04	0.03	-0.08	-0.06	-0.04	-0.03	-0.17	-0.12	-0.08	-0.06
500	0.06	0.03	0.01	0.01	-0.06	-0.02	-0.01	-0.01	-0.12	-0.05	-0.02	-0.01
	RMSE											
100	0.35	0.39	0.38	0.36	0.35	0.37	0.39	0.37	0.51	0.56	0.56	0.52
200	0.23	0.23	0.21	0.21	0.24	0.24	0.22	0.23	0.36	0.34	0.30	0.31
500	0.14	0.12	0.11	0.11	0.14	0.13	0.12	0.12	0.22	0.18	0.16	0.17

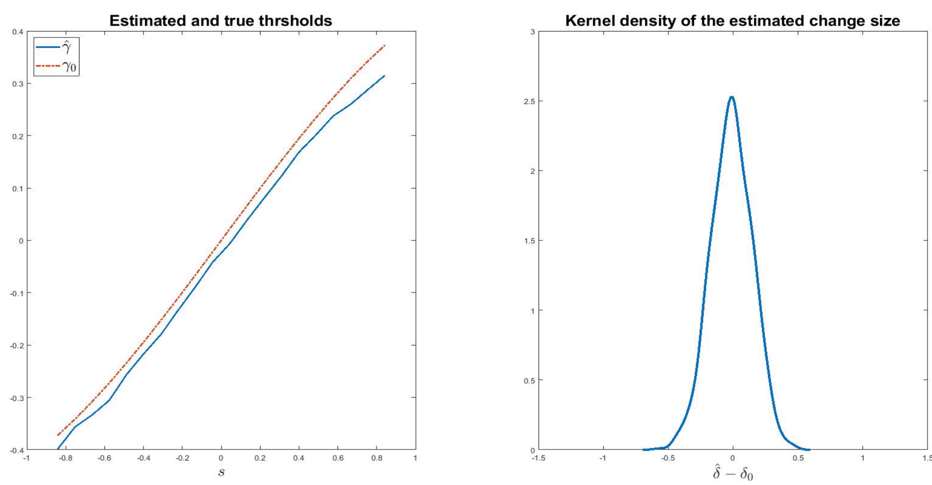
Note: Entries are bias and root mean squared error (RMSE) of the proposed two-step estimator for β_{20}, $\beta_{20} + \delta_{20}$, and δ_{20}. Data are generated from (18) with $\gamma_0(s) = \sin(s)/2$, where the dependence structure is given in (19) with $\rho = 0.5$ and $m = 3$. The results are based on 1000 simulations.

Table S.4: Coverage Prob. of the Confidence Intervals

$n \backslash \delta$	β_{20}				$\beta_{20}+\delta_{20}$				δ_{20}			
	1	2	3	4	1	2	3	4	1	2	3	4
	Coverage without small sample LRV adjustment											
100	0.83	0.86	0.88	0.89	0.85	0.87	0.89	0.87	0.84	0.86	0.89	0.89
200	0.86	0.90	0.93	0.94	0.89	0.90	0.93	0.92	0.84	0.90	0.92	0.93
500	0.87	0.93	0.94	0.93	0.89	0.93	0.94	0.93	0.85	0.94	0.95	0.93
	Coverage with small sample LRV adjustment											
100	0.91	0.94	0.94	0.94	0.92	0.93	0.94	0.95	0.91	0.92	0.95	0.95
200	0.93	0.95	0.96	0.98	0.93	0.96	0.97	0.96	0.92	0.96	0.97	0.97
500	0.93	0.97	0.97	0.97	0.93	0.95	0.97	0.97	0.91	0.97	0.97	0.97

Note: Entries are coverage probabilities of 95% confidence intervals for β_{20}, $\beta_{20}+\delta_{20}$, and δ_{20} with and without a small sample adjustment of the LRV estimator. Data are generated from (18) with $\gamma_0(s) = \sin(s)/2$, where the dependence structure is given in (19) with $\rho = 0.5$ and $m = 3$. The results are based on 1000 simulations.

Figure S.1: The Average of the Threshold Estimates and Kernel Density of Coefficient Estimates



Note: The left panel depicts the average of $\hat{\gamma}(s)$ and the right panel depicts the kernel density of $\hat{\delta}_2 - \delta_{20}$ from 1000 simulations. Data are generated from (18) with $\gamma_0(s) = \sin(s)/2$, where the dependence structure is given in (19) with $\rho = 0.5$ and $m = 3$.

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