

Supplementary Appendix to

“Bias in Dynamic Panel Models under Time Series Misspecification”

2012, *Journal of Econometrics*, Vol. 169:1, pp. 54-60

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1 Proof of Lemma A1

For $V_{i,t} = \sum_{j=0}^{\infty} A^j U_{i,t-j}$, we first derive the following lemma.

Lemma B1 *Under Assumptions E and S, for fixed T and t ,*

$$\mathbb{E} \left(V_{i,t-1} \sum_{s=1}^T V'_{i,s-1} \right) = (I_p - A)^{-1} (I_p - A^t) \Gamma_0 + \left\{ (I_p - A)^{-1} (I_p - A^{T-t}) A \Gamma_0 \right\}'.$$

Proof of Lemma B1 We derive that

$$\begin{aligned} \mathbb{E} \left(V_{i,t-1} \sum_{s=1}^T V'_{i,s-1} \right) &= \mathbb{E} \left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1} \right) \left(\sum_{s=1}^T \sum_{k=0}^{\infty} U'_{i,s-k-1} A'^k \right) \\ &= \sum_{s=1}^t \mathbb{E} \left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1} \sum_{k=0}^{\infty} U'_{i,s-k-1} A'^k \right) + \sum_{s=t+1}^T \mathbb{E} \left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1} \sum_{k=0}^{\infty} U'_{i,s-k-1} A'^k \right) \\ &= \sum_{s=1}^t \sum_{j=0}^{\infty} A^{j+t-s} \mathbb{E} (U_{i,s-j-1} U'_{i,s-j-1}) A'^j + \sum_{s=1}^{T-t} \sum_{j=0}^{\infty} A^j \mathbb{E} (U_{i,s-j-1} U'_{i,s-j-1}) A'^{j+s} \\ &= \sum_{j=1}^t A^{j-1} \Gamma_0 + \Gamma_0 \sum_{j=1}^{T-t} A'^j \\ &= (I_p - A)^{-1} (I_p - A^t) \Gamma_0 + \left\{ (I_p - A)^{-1} (I_p - A^{T-t}) A \Gamma_0 \right\}' \end{aligned}$$

since Γ_0 is symmetric. ■

Proof of Lemma A1 Since $V_{i,t-1}^0 = V_{i,t-1} - V_{i,\cdot-1}$,

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1}^0 U_{i,t}^{0'} &= \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T \left(V_{i,t-1} - \frac{1}{T} \sum_{s=1}^T V_{i,s-1} \right) \left(U_{i,t} - \frac{1}{T} \sum_{s=1}^T U_{i,s} \right) \\ &= \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1} U_{i,t}' - \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1} \sum_{s=1}^T U_{i,s}' \\ &= C_{11}(N, T) - C_{12}(N, T), \end{aligned}$$

where

$$\text{plim}_{N \rightarrow \infty} C_{11}(N, T) = \mathbb{E} \sum_{t=1}^T \left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1} U'_{i,t} \right) = 0$$

and

$$\begin{aligned} \text{plim}_{N \rightarrow \infty} C_{12}(N, T) &= \frac{1}{T} \mathbb{E} \sum_{t=1}^T \left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1} \sum_{s=1}^T U'_{i,s} \right) = \frac{1}{T} \sum_{t=2}^T \sum_{j=0}^{t-2} A^j \mathbb{E} (U_{i,t-j-1} U'_{i,t-j-1}) \\ &= \sigma^2 (I_p - A)^{-1} \left(I_p - \frac{1}{T} (I_p - A)^{-1} (I_p - A^T) \right) M, \end{aligned}$$

with $M = e_p e'_p$. Second, we have

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1}^0 V_{i,t-1}^{0'} &= \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T \left(V_{i,t-1} - \frac{1}{T} \sum_{s=1}^T V_{i,s-1} \right) \left(V'_{i,t-1} - \frac{1}{T} \sum_{s=1}^T V'_{i,s-1} \right) \\ &= \frac{1}{N} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1} V'_{i,t-1} - \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1} \left(\sum_{s=1}^T V'_{i,s-1} \right) \\ &= C_{21}(N, T) - C_{22}(N, T), \end{aligned}$$

where

$$\begin{aligned} \text{plim}_{N \rightarrow \infty} C_{21}(N, T) &= \sum_{t=1}^T \mathbb{E} \left(\sum_{j=0}^{\infty} A^j U_{i,t-j-1} \right) \left(\sum_{k=0}^{\infty} U'_{i,t-k-1} A'^k \right) \\ &= \sum_{t=1}^T \sum_{j=0}^{\infty} A^j \mathbb{E} (U_{i,t-j-1} U'_{i,t-j-1}) A'^j = T \Gamma_0 \end{aligned}$$

and using Lemma B1

$$\begin{aligned} \text{plim}_{N \rightarrow \infty} C_{22}(N, T) &= \frac{1}{T} \sum_{t=1}^T (I_p - A)^{-1} (I_p - A^t) \Gamma_0 + \frac{1}{T} \sum_{t=1}^T \left\{ (I_p - A)^{-1} (I_p - A^{(T-t)}) A \Gamma_0 \right\}' \\ &= (I_p - A)^{-1} \left(I_p - \frac{1}{T} (I_p - A)^{-1} (I_p - A^T) A \right) \Gamma_0 \\ &\quad + \left\{ (I_p - A)^{-1} \left(I_p - \frac{1}{T} (I_p - A)^{-1} (I_p - A^T) \right) A \Gamma_0 \right\}' . \blacksquare \end{aligned}$$

2 Proof of Lemma A2

We let $\hat{\Gamma}_j = (1/NT) \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1}^0 V_{i,t-1+j}^{0'}$ for $j = 0, 1$. The first derive the following lemmas.

Lemma B2 For large T ,

$$\begin{aligned} G_{01} &= (1/T) \Omega + O(T^{-2}), \\ G_{11} &= (1/T) \Omega A' + O(T^{-2}), \\ G_{12} &= (\sigma^2/T) (I_p - A)^{-1} (I_p + O(T^{-1})) M = (\sigma^2/T) (I_p - A)^{-1} M + O(T^{-2}). \end{aligned}$$

Proof of Lemma B2 Since Assumptions S and E imply that $H_T = O(T^{-1})$ and A is bounded,

$$\begin{aligned} G'_{01} &= (1/T) \left[(I_p - A)^{-1} (I_p + O(T^{-1})) \Gamma_0 + \left\{ (I_p - A)^{-1} (I_p + O(T^{-1})) A \Gamma_0 \right\}' \right] \\ &= (1/T) \left[\Gamma_0 (I_p - A')^{-1} + (I_p - A)^{-1} A \Gamma_0 \right] + O(T^{-2}) \\ &= (1/T) [\Delta + \Lambda'] + O(T^{-2}) \\ &= (1/T) \Omega + O(T^{-2}), \end{aligned}$$

where $\Delta = \sum_{j=0}^{\infty} \Gamma_j = \sum_{j=0}^{\infty} \Gamma_0 A'^j = \Gamma_0 (I_p - A')^{-1}$ and $\Lambda' = (\sum_{j=1}^{\infty} \Gamma_j)' = (\Gamma_1 (I_p - A')^{-1})' = (I_p - A)^{-1} A \Gamma_0$. Note that Ω is symmetric. Similarly, since $G_{11} = G_{01} \Gamma_0^{-1} \Gamma_1 = G_{01} A'$ and A is bounded, the remaining results readily follows. ■

Lemma B3 Under Assumptions E, S, I and NT,

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=1}^T \text{vec}(V_{i,t-1} U'_{i,t}) \rightarrow_d \mathcal{N}(0, \sigma^2 (M \otimes \Gamma_0)) \quad \text{as } N, T \rightarrow \infty \text{ jointly.}$$

Proof of Lemma B3 We first check the generalized Lindeberg-Feller condition for joint (i.e., double indexed) asymptotic normality as in Theorem 2 of Phillips and Moon (1999). As discussed in Hahn and Kuersteiner (2002), a sufficient condition for the theorem to hold is that $\mathbb{E} \left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \varphi' V_{i,t-1} u_{i,t} \right)^4 \right]$ is bounded uniformly in i and T for all nonrandom vector $\varphi \in \mathbb{R}^p$ such that $\varphi' \varphi = 1$. Recall that $U_{i,t} = e u_{i,t} = (u_{i,t}, 0, \dots, 0)'$, and thus we only need to look at the moment condition of the first column of $V_{i,t-1} U'_{i,t}$ to check if this condition holds. We let $z_{i,t} = \varphi' V_{i,t-1} u_{i,t} = \sum_{j=0}^{\infty} \varphi' A^j U_{i,t-1-j} u_{i,t}$, then $\mathbb{E} z_{i,t} = 0$ and $\mathbb{E} z_{i,t} z_{i,s} = 0$ for $t \neq s$ since $u_{i,t}$ is *i.i.d.* It follows that $\mathbb{E} \left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^T z_{i,t} \right)^4 \right] = (1/T^2) \sum_{t=1}^T \mathbb{E} z_{i,t}^4 + (1/T^2) \sum_{t \neq s} \mathbb{E} z_{i,t}^2 \mathbb{E} z_{i,s}^2$. We observe that

$$\begin{aligned} \mathbb{E} z_{i,t}^2 &= \mathbb{E} \left(\sum_{j=0}^{\infty} \varphi' A^j U_{i,t-1-j} u_{i,t} \right) \left(\sum_{k=0}^{\infty} \varphi' A^k U_{i,t-1-k} u_{i,t} \right)' \\ &= (\mathbb{E} u_{i,t}^2) \mathbb{E} \left(\varphi' \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A^j U_{i,t-1-j} U'_{i,t-1-k} A^k \varphi \right) \\ &= \sigma^2 \varphi' \mathbb{E} \left(\sum_{j=0}^{\infty} A^j U_{i,t-1-j} U'_{i,t-1-j} A^j \right) \varphi = \sigma^2 \varphi' \Gamma_0 \varphi < \infty \end{aligned}$$

since $\mathbb{E}A^jU_{i,t-1-j}U'_{i,t-1-k}A'^k = 0$ if $j \neq k$; and Γ_0 if $j = k$. Moreover,

$$\mathbb{E}z_{i,t}^4 = (\mathbb{E}u_{i,t}^4) \mathbb{E} \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \varphi' A^j U_{i,t-1-j} U'_{i,t-1-k} A'^k \varphi \varphi' A^l U_{i,t-1-l} U'_{i,t-1-m} A'^m \varphi \right),$$

and if we define a random variable $w_j = \varphi' A^j U_{i,t-1-j}$ with $\mathbb{E}w_j = 0$ then

$$\begin{aligned} & \mathbb{E} \left(\varphi' A^j U_{i,t-1-j} U'_{i,t-1-k} A'^k \varphi \varphi' A^l U_{i,t-1-l} U'_{i,t-1-m} A'^m \varphi \right) \\ &= \mathbb{E} (w_j w_k w_l w_m) \\ &= \mathbb{E} (w_j w_k) \mathbb{E} (w_l w_m) + \mathbb{E} (w_j w_l) \mathbb{E} (w_k w_m) + \mathbb{E} (w_j w_m) \mathbb{E} (w_k w_l) + \mathcal{K} (w_j, w_k, w_l, w_m), \end{aligned}$$

where $\mathcal{K} (w_j, w_k, w_l, w_m)$ is the fourth order cumulant of w_j . Note that similarly as $\mathbb{E}z_{i,t}^2$, we have $\mathbb{E} (w_j w_k) = \sigma^2 \varphi' A^j M A'^j \varphi$ if $j = k$ and zero otherwise. Therefore,

$$\begin{aligned} |\mathbb{E}z_{i,t}^4| &\leq |\mathbb{E}u_{i,t}^4| \left\{ 3 \left(\varphi' \left| \sigma^2 \sum_{j=0}^{\infty} A^j M A'^j \right| \varphi \right)^2 + \sum_{j,k,l,m=0}^{\infty} |\mathcal{K} (w_j, w_j, w_j, w_j)| \right\} \\ &\leq 3 |\mathbb{E}u_{i,t}^4| (\varphi' \Gamma_0 \varphi)^2 \\ &+ |\mathbb{E}u_{i,t}^4| \sum_{r_1, \dots, r_4=1}^p \varphi_{r_1} \varphi_{r_2} \varphi_{r_3} \varphi_{r_4} \sum_{j,k,l,m=0}^{\infty} |\mathcal{K}_{r_1, \dots, r_4} (A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j})| \\ &< \infty \end{aligned}$$

since $\sum_{j,k,l,m=0}^{\infty} |\mathcal{K}_{r_1, \dots, r_4} (A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j}, A^j U_{i,t-1-j})| < \infty$, which is by Lemma 1 in Hahn and Kuersteiner (2002) for $u_{i,t}$ is i.i.d. and it has finite eighth moments (Assumption E). As a consequence, we have $\mathbb{E}((1/\sqrt{T}) \sum_{t=1}^T z_{i,t})^4 < \infty$, which provides a sufficient condition for the generalized Lindeberg-Feller condition (Phillips and Moon, 1999, Theorem 2). For the variance term, note that

$$\begin{aligned} \mathbb{E} \left[\text{vec} (V_{i,t-1} U'_{i,t}) (\text{vec} (V_{i,t-1} U'_{i,t}))' \right] &= \mathbb{E} [(I_p \otimes V_{i,t-1}) \text{vec} (U'_{i,t})] [(I_p \otimes V_{i,t-1}) \text{vec} (U'_{i,t})]' \\ &= \mathbb{E} (I_p \otimes V_{i,t-1}) U_{i,t} U'_{i,t} (I_p \otimes V'_{i,t-1}) \\ &= \mathbb{E} [(I_p \otimes V_{i,t-1}) (\mathbb{E}_{t-1} U_{i,t} U'_{i,t}) (I_p \otimes V'_{i,t-1})] \\ &= \sigma^2 \mathbb{E} [(I_p \otimes V_{i,t-1}) e e' (I_p \otimes V'_{i,t-1})] \\ &= \sigma^2 \mathbb{E} [(I_p \otimes V_{i,t-1}) (e \otimes 1)] [(e' \otimes 1) (I_p \otimes V'_{i,t-1})] \\ &= \sigma^2 \mathbb{E} (e \otimes V_{i,t-1}) (e' \otimes V'_{i,t-1}) \\ &= \sigma^2 (M \otimes \Gamma_0). \end{aligned}$$

$\mathbb{E}_{t-1} (\cdot)$ is the conditional expectation given $\{u_{i,s} : s \leq t, \text{ for all } i\}$ and $\mathbb{E}_{t-1} (U_{i,t} U'_{i,t}) = \mathbb{E} (U_{i,t} U'_{i,t})$ because $u_{i,t}$ is i.i.d. Note that $\text{vec}(U_{i,t}) = \text{vec}(U'_{i,t}) = U_{i,t}$ and we use the identity $\text{vec}(BC) = (I \otimes B) \text{vec}(C)$. ■

Proof of Lemma A2 We observe that

$$\begin{aligned} \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1}^0 U_{i,t}^{0'} &= \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=1}^T V_{i,t-1} U'_{i,t} - \sqrt{\frac{N}{T}} \cdot \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T V_{i,t-1} U'_{i,s} \\ &= C_{31}(N, T) - C_{32}(N, T). \end{aligned}$$

By Lemma B3, $\text{vec}(C_{31}(N, T)) \rightarrow_d \mathcal{N}(0, \sigma^2 (M \otimes \Gamma_0))$ as $N, T \rightarrow \infty$. For $C_{32}(N, T)$, it can be shown that $C_{32}(N, T) \rightarrow_p \sqrt{\kappa} \sigma^2 (I_p - A)^{-1} M$ as $N, T \rightarrow \infty$. This result requires integrability of $(1/T) \sum_{t=1}^T \sum_{s=1}^T V_{i,t-1} U'_{i,s}$ to satisfy the conditions for joint probability limit in Phillips and Moon (1999, Theorem 1), which indeed directly follows from $C_{12}(N, T)$ in the proof of Lemma A1.¹ Therefore,

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=1}^T \text{vec}(V_{i,t-1}^0 U_{i,t}^{0'}) \rightarrow_d \mathcal{N}\left(-\sqrt{\kappa} \sigma^2 \text{vec}\left((I_p - A)^{-1} M\right), \sigma^2 (M \otimes \Gamma_0)\right). \blacksquare$$

References

- [1] HAHN, J. AND G. KUERSTEINER (2002). Asymptotically unbiased inference for a dynamic panel model with fixed effects, *Econometrica*, 70, 1639-1657.
- [2] PHILLIPS, P.C.B. AND H.R. MOON (1999). Linear Regression Limit Theory for Nonstationary Panel Data, *Econometrica*, 67, 1057-1111.

¹In the proof of Theorem 2, we state that $\sqrt{N/T} \cdot T(\widehat{\Gamma}_0^{Y|X} - \Gamma_0^{Y|X})A' \rightarrow_p -\sqrt{\kappa}\Omega A'$ as $N, T \rightarrow \infty$. In order to prove this result, however, we also need to check the conditions for joint probability limit as in Theorem 1 of Phillips and Moon (1999) since we are considering probability limit of double indexed process. In this case, the integrability of $\sum_{t=1}^T (V_{i,t-1}^0 V_{i,t-1}^{0'} - \Gamma_0)$ is to be verified. (From Lemma A1, all other conditions are obviously satisfied.) Note that

$$\begin{aligned} \left\| \mathbb{E} \sum_{t=1}^T (V_{i,t-1}^0 V_{i,t-1}^{0'} - \Gamma_0) \right\| &\leq \left\| \sum_{t=1}^T \mathbb{E} (V_{i,t-1} V_{i,t-1}' - \Gamma_0) \right\| + \left\| \frac{1}{T} \mathbb{E} \left(\sum_{t=1}^T V_{i,t-1} \right) \left(\sum_{t=1}^T V_{i,t-1}' \right) \right\| \\ &\leq \left\| (I_p - A)^{-1} \right\| \{ \|I_p - H_T A\| + \|(I_p - H_T) A\| \} \|\Gamma_0\| < \infty \end{aligned}$$

because Assumptions S and E imply that A , $(I_p - A)^{-1}$, TH_T and Γ_0 are all bounded for large T since $\|(I_p - A)^{-1}\| \leq \sum_{j=0}^{\infty} \|A^j\| < \infty$ and $\|TH_T\| \leq \sum_{t=1}^T \|A^{t-1}\| \leq \sum_{j=0}^{\infty} \|A^j\| < \infty$.